SPHERICAL FUNCTIONS ASSOCIATED TO THE THREE DIMENSIONAL SPHERE

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ABSTRACT. In this paper, we determine all irreducible spherical functions Φ of any K-type associated to the pair $(G, K) = (\mathrm{SO}(4), \mathrm{SO}(3))$. This is accomplished by associating to Φ a vector valued function $H = H(u)$ of a real variable u, which is analytic at $u = 0$ and whose components are solutions of two coupled systems of ordinary differential equations. By an appropriate conjugation involving Hahn polynomials we uncouple one of the systems. Then this is taken to an uncoupled system of hypergeometric equations, leading to a vector valued solution $P = P(u)$ whose entries are Gegenbauer's polynomials. Afterward, we identify those simultaneous solutions and use the representation theory of SO(4) to characterize all irreducible spherical functions. The functions $P = P(u)$ corresponding to the irreducible spherical functions of a fixed K-type π_ℓ are appropriately packaged into a sequence of matrix valued polynomials $(P_w)_{w\geq 0}$ of size $(\ell+1)\times(\ell+1)$. Finally we proved that $\widetilde{P}_w = {P_0}^{-1} P_w$ is a sequence of matrix orthogonal polynomials with respect to a weight matrix W. Moreover we showed that W admits a second order symmetric hypergeometric operator $\widetilde{D}$ and a first order symmetric differential operator $\widetilde{E}$.

1. Introduction

The theory of spherical functions dates back to the classical papers of É. Cartan and H. Weyl; they showed that spherical harmonics arise in a natural way from the study of functions on the n-dimensional sphere $S^n = \mathrm{SO}(n+1)/\mathrm{SO}(n)$. The first general results in this direction were obtained in 1950 by Gelfand who considered zonal spherical functions of a Riemannian symmetric space G/K. In this case we have a polar decomposition $G = KAK$. When the abelian subgroup A is one dimensional the restrictions of zonal spherical functions to A can be identified with hypergeometric functions, providing a deep and fruitful connection between group representation theory and special functions. In particular when G is compact this gives a one to one correspondence between all zonal spherical functions of the symmetric pair (G, K) and a sequence of orthogonal polynomials.

In light of this remarkable background it is reasonable to look for an extension of the above results, by considering matrix valued spherical functions on G of a general K-type. This was accomplished for the first time in the case of the complex projective plane $P_2(\mathbb{C}) = \mathrm{SU}(3)/\mathrm{U}(2)$ in [GPT2]. This seminal work gave rise to a series of papers including [GPT1, GPT4, GPT6, PT1, PT2, PR], where one

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considers matrix valued spherical functions associated to a compact symmetric pair (G, K), arriving at sequences of matrix valued orthogonal polynomials of one real variable satisfying an explicit three term recursion relation, which are also eigenfunctions of a second order matrix differential operator (bispectral property). In particular in [G] the first example of a matrix weight function on the real line with a symmetric matrix second order differential operator was exhibited. Related results can be also found in [GPT5, PT11, GT, RT, RT2]. In [PT12] the irreducible spherical functions associated to the complex projective space $P_n(\mathbb{C})$ of a given K-type are encoded in a sequence of matrix valued orthogonal polynomials, which are given in terms of the matrix hypergeometric function. The three term recursion relation was obtained in [P] and the corresponding semi infinite matrix turns out to be stochastic. This unexpected result leads to the study of the random walk with this transition probability matrix, see [GPT7].

A different approach to find examples of classical matrix orthogonal polynomials can be found in [DG].

The present paper is an outgrowth of [Z] and we are presently working on the extension of the present results to the n-dimensional sphere and the n-dimensional real projective space.

Briefly the main results of this paper are the following. After a preliminary work developed along the first sections, in Section 5 we are able to explicitly describe the irreducible spherical functions of the symmetric pair $(\mathrm{SO}(4), \mathrm{SO}(3))$ of a fixed K-type, by a vector valued function $P = P(u)$ whose entries are certain Gegenbauer polynomials in a suitable variable u. This is accomplished by uncoupling a system of second order linear differential equations using a constant matrix of Hahn polynomials evaluated at 1, see Corollary 5.5.

In Section 7 it is established which are those vector valued polynomials $P = P(u)$ who correspond to irreducible spherical functions, and it is shown how to reconstruct the spherical functions out of them.

The aim of the last two sections is to build classical sequences of matrix valued orthogonal polynomials from our previous work. In Section 8 we defined a sequence of polynomial matrices P_w, $w \geq 0$, whose columns are the vector valued polynomials $P = P(u)$ corresponding to some specific irreducible spherical functions of the same K-type. Then we consider the sequence $\widetilde{P}_w = {P_0}^{-1} P_w$ and we prove that $(\widetilde{P}_w)_{w\geq 0}$ is a sequence of matrix orthogonal polynomials with respect to a weight function explicitly given in (60). Moreover the matrix differential operators $\widetilde{D}$ and $\widetilde{E}$ given in Theoren 8.3 satisfy $\widetilde{D}\widetilde{P}_w = \widetilde{P}_w \Lambda_w$ and $\widetilde{E}\widetilde{P}_w = \widetilde{P}_w M_w$, where the eigenvalue matrices Λ_w and M_w are real diagonal matrices. Thus $\widetilde{D}$ and $\widetilde{E}$ are symmetric with respect to W.

2. Preliminaries

2.1. **Spherical functions.**

Let G be a locally compact unimodular group and let K be a compact subgroup of G. Let $\hat{K}$ denote the set of all equivalence classes of complex finite dimensional irreducible representations of K; for each $\delta \in \hat{K}$, let ξ_δ denote the character of δ, $d(\delta)$ the degree of δ, i.e. the dimension of any representation in the class δ, and $\chi_\delta = d(\delta)\xi_\delta$. We shall choose once and for all the Haar measure dk on K normalized by $\int_K dk = 1$.

We shall denote by V a finite dimensional vector space over the field $\mathbb{C}$ of complex numbers and by $\mathrm{End}(V)$ the space of all linear transformations of V into V. Whenever we refer to a topology on such a vector space we shall be talking about the unique Hausdorff linear topology on it.

Definition 2.1. A spherical function Φ on G of type $\delta \in \hat{K}$ is a continuous function on G with values in $\mathrm{End}(V)$ such that

i) $\Phi(e) = I$ (I= identity transformation).
ii) $\Phi(x)\Phi(y) = \int_K \chi_\delta(k^{-1})\Phi(xky)\, dk$, for all $x, y \in G$.

The reader can find a number of general results in [Tir77] and [GV88]. For our purpose it is appropriate to recall the following facts.

Proposition 2.2. *(*[Tir77],[GV88]*) If $\Phi : G \longrightarrow \mathrm{End}(V)$ is a spherical function of type δ then:*

i) *$\Phi(kgk') = \Phi(k)\Phi(g)\Phi(k')$, for all $k, k' \in K$, $g \in G$.*
ii) *$k \mapsto \Phi(k)$ is a representation of K such that any irreducible subrepresentation belongs to δ.*

Concerning the definition let us point out that the spherical function Φ determines its type univocally (Proposition 2.2) and let us say that the number of times that δ occurs in the representation $k \mapsto \Phi(k)$ is called the *height* of Φ.

If G is a connected Lie group it is not difficult to prove that any spherical function $\Phi : G \longrightarrow \mathrm{End}(V)$ is differentiable (C^∞), and moreover that it is analytic. Let $D(G)$ denote the algebra of all left invariant differential operators on G and let $D(G)^K$ denote the subalgebra of all operators in $D(G)$ which are invariant under all right translations by elements in K.

In the following proposition (V, π) will be a finite dimensional representation of K such that any irreducible subrepresentation belongs to the same class $\delta \in \hat{K}$.

Proposition 2.3. *(*[Tir77],[GV88]*) A function $\Phi : G \longrightarrow \mathrm{End}(V)$ is a spherical function of type δ if and only if*

i) *Φ is analytic.*
ii) *$\Phi(k_1 g k_2) = \pi(k_1)\Phi(g)\pi(k_2)$, for all $k_1, k_2 \in K$, $g \in G$, and $\Phi(e) = I$.*
iii) *$[D\Phi](g) = \Phi(g)[D\Phi](e)$, for all $D \in D(G)^K$, $g \in G$.*

Moreover, we have that the eigenvalues $[D\Phi](e)$, $D \in D(G)^K$, characterize the spherical functions Φ as it is stated in the following proposition.

Proposition 2.4. *(*[Tir77],[GV88]*) Let $\Phi, \Psi : G \longrightarrow \mathrm{End}(V)$ be two spherical functions on a connected Lie group G of the same type $\delta \in K$. Then $\Phi = \Psi$ if and only if $(D\Phi)(e) = (D\Psi)(e)$ for all $D \in D(G)^K$.*

Let us observe that if $\Phi : G \longrightarrow \mathrm{End}(V)$ is a spherical function then $\Phi : D \mapsto [D\Phi](e)$ maps $D(G)^K$ into $\mathrm{End}_K(V)$ ($\mathrm{End}_K(V)$ denotes the space of all linear maps of V into V which commutes with $\pi(k)$ for all $k \in K$) defining a finite dimensional representation of the associative algebra $D(G)^K$. Moreover the spherical function is irreducible if and only if the representation $\Phi : D(G)^K \longrightarrow \mathrm{End}_K(V)$ is irreducible. As a consequence of this we have:

Proposition 2.5. *(*[Tir77],[GV88]*) The following properties are equivalent:*

i) *$D(G)^K$ is commutative.*

ii) *Every irreducible spherical function of* (G, K) *is of height one.*

In this paper the pair (G, K) is $(\mathrm{SO}(4), \mathrm{SO}(3))$. Then it is known that $D(G)^K$ is an abelian algebra; moreover $D(G)^K$ is isomorphic to $D(G)^G \otimes D(K)^K$ (cf. [Coo75], [Kno86]), where $D(G)^G$ (resp. $D(K)^K$) denotes the subalgebra of all operators in $D(G)$ (resp. $D(K)$) which are invariant under all right translations by elements in G (resp. K).

Now, in our case we have that $D(G)^G$ is a polynomial algebra in two algebraically independent generators. This is because the Lie algebra of G is $\mathfrak{so}(4) \simeq \mathfrak{so}(3) \oplus \mathfrak{so}(3)$, hence if Δ_1 and Δ_2 are the Casimirs of the corresponding $\mathfrak{so}(3)$ we have that Δ_1 and Δ_2 generate $D(G)^G$.

The first consequence of this is that all irreducible spherical functions of our pair (G, K) are of height one. The second consequence is that to find all irreducible spherical functions of type $\delta \in \hat{K}$ is equivalent to take any irreducible representation (V, π) of K in the class δ and to determine all analytic functions $\Phi : G \longrightarrow \mathrm{End}(V)$ such that

i) $\Phi(k_1 g k_2) = \pi(k_1)\Phi(g)\pi(k_2)$, for all $k_1, k_2 \in K$, $g \in G$, and $\Phi(e) = I$.

ii) $[\Delta_1\Phi](g) = \tilde{\lambda}\Phi(g)$, $[\Delta_2\Phi](g) = \tilde{\mu}\Phi(g)$ for all $g \in G$ and for some $\tilde{\lambda}, \tilde{\mu} \in \mathbb{C}$.

A particular choice of these operators Δ_1 and Δ_2 is given in (2).

Spherical functions of type δ arise in a natural way upon considering representations of G. If $g \mapsto U(g)$ is a continuous representation of G, say on a finite dimensional vector space E, then

$$P_\delta = \int_K \chi_\delta(k^{-1})U(k)\, dk$$

is a projection of E onto $P_\delta E = E(\delta)$. The function $\Phi : G \longrightarrow \mathrm{End}(E(\delta))$ defined by

$$\Phi(g)a = P_\delta U(g)a, \quad g \in G,\ a \in E(\delta),$$

is a spherical function of type δ. In fact, if $a \in E(\delta)$ we have

$$\begin{aligned}\Phi(x)\Phi(y)a = P_\delta U(x) P_\delta U(y) a &= \int_K \chi_\delta(k^{-1}) P_\delta U(x)U(k)U(y)a\, dk \\ &= \left(\int_K \chi_\delta(k^{-1})\Phi(xky)\, dk\right) a.\end{aligned}$$

If the representation $g \mapsto U(g)$ is irreducible then the associated spherical function Φ is also irreducible. Conversely, any irreducible spherical function on a compact group G arises in this way from a finite dimensional irreducible representation of G.

2.2. The groups G and K.

The three dimensional sphere S^3 can be realized as the homogeneous space G/K, where $G = \mathrm{SO}(4)$ and $K = \mathrm{SO}(3)$, where as usual we identify $\mathrm{SO}(3)$ as a subgroup of $\mathrm{SO}(4)$: for every k in K, let $k = \left(\begin{smallmatrix} k & 0 \\ 0 & 1 \end{smallmatrix}\right) \in G$.

It is well known that there exists a double covering Lie homomorphism $\mathrm{SO}(4) \longrightarrow \mathrm{SO}(3) \times \mathrm{SO}(3)$, in particular $\mathfrak{so}(4) \simeq \mathfrak{so}(3) \oplus \mathfrak{so}(3)$. Explicitly it is obtained in the following way: Let $q : \mathrm{SO}(4) \longrightarrow \mathrm{GL}\big(\Lambda^2(\mathbb{R}^4)\big)$ be the Lie homomorphism defined by

$$q(g)(e_i \wedge e_j) = g(e_i) \wedge g(e_j)\,, \qquad g \in \mathrm{SO}(4)\,, \qquad 1 \le i < j \le 4,$$

where $\{e_j\}_{j=1}^4$ is the canonical basis of $\mathbb{R}^4$. Let $\dot{q} : \mathfrak{so}(4) \longrightarrow \mathfrak{gl}\big(\Lambda^2(\mathbb{R}^4)\big)$ denote the corresponding differential homomorphism.

We observe that $\Lambda^2(\mathbb{R}^4)$ is reducible as G-module. In fact we have the following decomposition into irreducible G-modules, $\Lambda^2(\mathbb{R}^4) = V_1 \oplus V_2$, where

$$V_1 = \operatorname{span}\{e_1 \wedge e_4 + e_2 \wedge e_3, e_1 \wedge e_3 - e_2 \wedge e_4, -e_1 \wedge e_2 - e_3 \wedge e_4\},$$

$$V_2 = \operatorname{span}\{e_1 \wedge e_4 - e_2 \wedge e_3, e_1 \wedge e_3 + e_2 \wedge e_4, -e_1 \wedge e_2 + e_3 \wedge e_4\}.$$

Let P_1 and P_2 be the canonical projections onto the subspaces V_1 and V_2, respectively. The functions defined by

$$a(g) = P_1\, q(g)_{|V_1}, \qquad b(g) = P_2\, q(g)_{|V_2},$$

are Lie homomorphisms from SO(4) onto SO$(V_1) \simeq$ SO(3) and SO$(V_2) \simeq$ SO(3), respectively. Therefore in an appropriate basis we have, for each $g \in$ SO(4) and for all $X \in \mathfrak{so}(4)$

$$q(g) = \begin{pmatrix} a(g) & 0 \\ 0 & b(g) \end{pmatrix}, \qquad \dot{q}(X) = \begin{pmatrix} \dot{a}(X) & 0 \\ 0 & \dot{b}(X) \end{pmatrix}. \tag{1}$$

Hence, we can consider q as a homomorphism from SO(4) onto SO(3) $\times$ SO(3) with kernel $\{I, -I\}$. Besides it can be proved that $a(g) = b(g)$ if and only if $g \in K$.

2.3. **The Lie algebra structure.**

A basis of $\mathfrak{g} = \mathfrak{so}(4)$ over $\mathbb{R}$ is given by

$$Y_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad Y_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad Y_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$Y_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}, \quad Y_5 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \quad Y_6 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}.$$

Consider the following vectors

$$Z_1 = \frac{1}{2}\left(Y_3 + Y_4\right), \quad Z_2 = \frac{1}{2}\left(Y_2 - Y_5\right), \quad Z_3 = \frac{1}{2}\left(Y_1 + Y_6\right),$$

$$Z_4 = \frac{1}{2}\left(Y_3 - Y_4\right), \quad Z_5 = \frac{1}{2}\left(Y_2 + Y_5\right), \quad Z_6 = \frac{1}{2}\left(Y_1 - Y_6\right).$$

It can be proved that these vectors define a basis of $\mathfrak{so}(4)$ adapted to the decomposition $\mathfrak{so}(4) \simeq \mathfrak{so}(3) \oplus \mathfrak{so}(3)$, i.e. $\{Z_4, Z_5, Z_6\}$ is a basis of the first summand and $\{Z_1, Z_2, Z_3\}$ is a basis of the second one.

The algebra $D(G)^G$ is generated by the algebraically independent elements

$$\Delta_1 = -Z_4^2 - Z_5^2 - Z_6^2\,, \qquad \Delta_2 = -Z_1^2 - Z_2^2 - Z_3^2, \tag{2}$$

which are the Casimirs of the the first and the second $\mathfrak{so}(3)$ respectively. The Casimir of K will be denoted by Δ_K, and it is given by $-Y_1^2 - Y_2^2 - Y_3^2$.

The complexification of $\mathfrak{k}$, is isomorphic to $\mathfrak{sl}(2, \mathbb{C})$. If we define

$$e = \begin{pmatrix} 0 & i & -1 \\ -i & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad f = \begin{pmatrix} 0 & i & 1 \\ -i & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \qquad h = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -2i \\ 0 & 2i & 0 \end{pmatrix} \tag{3}$$

then we have that $\{e, f, h\}$ is an s-triple in $\mathfrak{k}_{\mathbb{C}}$, i.e.

$$[e, f] = h, \qquad [h, e] = 2e, \qquad [h, f] = -2f.$$

We take as a Cartan subalgebra $\mathfrak{h}_{\mathbb{C}}$ of $\mathfrak{so}(4,\mathbb{C})$ the complexification of the maximal abelian subalgebra of $\mathfrak{so}(4)$ of all matrices of the form

$$H = \begin{pmatrix} 0 & x_1 & 0 & 0 \\ -x_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_2 \\ 0 & 0 & -x_2 & 0 \end{pmatrix}.$$

Let $\varepsilon_j \in \mathfrak{h}_{\mathbb{C}}^*$ be given by $\varepsilon_j(H) = -ix_j$ for $j = 1, 2$. Then

$$\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}) = \{\, \pm(\varepsilon_1 \pm \varepsilon_2) : \varepsilon_1, \varepsilon_2 \in \mathfrak{h}_{\mathbb{C}}^* \,\},$$

and we choose as positive roots those in the set $\Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}) = \{\varepsilon_1 - \varepsilon_2, \varepsilon_1 + \varepsilon_2\}$.

We define

$$\begin{aligned} X_{\varepsilon_1+\varepsilon_2} &= \begin{pmatrix} 0 & 0 & 1 & -i \\ 0 & 0 & -i & -1 \\ -1 & i & 0 & 0 \\ i & 1 & 0 & 0 \end{pmatrix}, \quad X_{\varepsilon_1-\varepsilon_2} = \begin{pmatrix} 0 & 0 & 1 & i \\ 0 & 0 & -i & 1 \\ -1 & i & 0 & 0 \\ -i & -1 & 0 & 0 \end{pmatrix}, \\ X_{-\varepsilon_1+\varepsilon_2} &= \begin{pmatrix} 0 & 0 & 1 & -i \\ 0 & 0 & i & 1 \\ -1 & -i & 0 & 0 \\ i & -1 & 0 & 0 \end{pmatrix}, X_{-\varepsilon_1-\varepsilon_2} = \begin{pmatrix} 0 & 0 & 1 & i \\ 0 & 0 & i & -1 \\ -1 & -i & 0 & 0 \\ -i & 1 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{4}$$

Then, for every H in $\mathfrak{h}_{\mathbb{C}}$ we get that

$$[H, X_{\pm(\varepsilon_1 \pm \varepsilon_2)}] = \pm(\varepsilon_1 \pm \varepsilon_2)(H) X_{\pm(\varepsilon_1 \pm \varepsilon_2)}.$$

Hence, each $X_{\pm(\varepsilon_1 \pm \varepsilon_2)}$ belongs to the root-space $\mathfrak{g}_{\pm(\varepsilon_1 \pm \varepsilon_2)}$.

Then, in terms of the root structure of $\mathfrak{so}(4,\mathbb{C})$, Δ_1 and Δ_2 become

$$\begin{aligned} \Delta_1 &= -Z_6^2 + iZ_6 - (Z_5 + iZ_4)(Z_5 - iZ_4), \\ \Delta_2 &= -Z_3^2 + iZ_3 - (Z_2 + iZ_1)(Z_2 - iZ_1). \end{aligned} \tag{5}$$

We observe that $(Z_5 - iZ_4) = X_{\varepsilon_1-\varepsilon_2} \in \mathfrak{g}_{\varepsilon_1-\varepsilon_2}$ and $(Z_2 - iZ_1) = X_{\varepsilon_1+\varepsilon_2} \in \mathfrak{g}_{\varepsilon_1+\varepsilon_2}$ and $Z_3, Z_6 \in \mathfrak{h}_{\mathbb{C}}$.

2.4. **Irreducible representations of G and K.**

Let us first consider SU(2). It is well known that the irreducible finite dimensional representations of SU(2) are, up to equivalence, $(\pi_\ell, V_\ell)_{\ell\geq 0}$, where V_ℓ is the complex vector space of all polynomial functions in two complex variables z_1 and z_2 homogeneous of degree ℓ, and π_ℓ is defined by

$$\pi_\ell \begin{pmatrix} a & b \\ c & d \end{pmatrix} P \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = P\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \right), \qquad \text{for } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SU}(2).$$

Hence, since there is a Lie homomorphism of SU(2) onto SO(3) with kernel $\{\pm I\}$, the irreducible representations of SO(3) corresponds to those representations π_ℓ of SU(2) with $\ell \in 2\mathbb{Z}_{\geq 0}$. Therefore we have $\hat{\mathrm{SO}}(3) = \{[\pi_\ell]\}_{\ell \in 2\mathbb{Z}_{\geq 0}}$, even more, if $\pi = \pi_\ell$ is any such irreducible representation of SO(3), it is well known (see [Hum72], page 32) that there exists a basis $\mathcal{B} = \{v_j\}_{j=0}^{\ell}$ of V_π such that the corresponding representation $\dot\pi$ of the complexification of $\mathfrak{so}(3)$ is given by

$$\begin{aligned} \dot\pi(h)v_j &= (\ell - 2j)v_j, \\ \dot\pi(e)v_j &= (\ell - j + 1)v_{j-1}, \quad (v_{-1} = 0), \\ \dot\pi(f)v_j &= (j+1)v_{j+1}, \quad (v_{\ell+1} = 0). \end{aligned}$$

It is known (see [VK], page 362) that an irreducible representation $\tau \in \hat{\mathrm{SO}}(4)$ has highest weight of the form $\eta = m_1 e_1 + m_2 e_2$, where m_1 and m_2 are integers such that $m_1 \geq |m_2|$. Moreover, the representation $\tau = \tau_{(m_1,m_2)}$, restricted to SO(3), contains the representation π_ℓ if and only if

$$m_1 \geq \tfrac{\ell}{2} \geq |m_2|.$$

2.5. **K-orbits in G/K.**

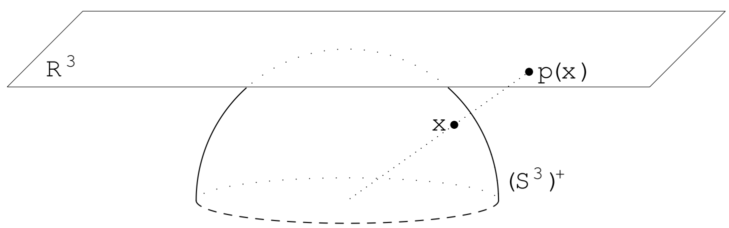
The group $G = \mathrm{SO}(4)$ acts in a natural way in the sphere S^3. This action is transitive and K is the isotropy subgroup of the north pole $e_4 = (0,0,0,1) \in S^3$. Therefore $S^3 \simeq G/K$. Moreover the G-action on S^3 corresponds to the action induced by left multiplication on G/K.

In the north hemisphere of S^3

$$(S^3)^+ = \left\{ x = (x_1, x_2, x_3, x_4) \in S^3 : x_4 > 0 \right\},$$

we will consider the coordinate system $p : (S^3)^+ \longrightarrow \mathbb{R}^3$ given by

$$p(x) = \left(\frac{x_1}{x_4}, \frac{x_2}{x_4}, \frac{x_3}{x_4} \right) = (y_1, y_2, y_3). \tag{6}$$



The coordinate map p carries the K-orbits in $(S^3)^+$ into the K-orbits in $\mathbb{R}^3$ which are the spheres

$$S_r = \left\{ (y_1, y_2, y_3) \in \mathbb{R}^3 : ||y||^2 = |y_1|^2 + |y_2|^2 + |y_3|^2 = r^2 \right\}, \qquad 0 \leq r < \infty.$$

In each orbit S_r we choose as a representative the point $(r,0,0) \in \mathbb{R}^3$ with $0 \leq r < \infty$. Then the interval $[0,\infty)$ parameterizes the set of K-orbits of $\mathbb{R}^3$.

2.6. **The auxiliary function Φ_π.**

As in [GPT2] to determine all irreducible spherical functions Φ of type $\pi = \pi_\ell \in \hat{K}$ an auxiliary function $\Phi_\pi : G \longrightarrow \mathrm{End}(V_\pi)$ is introduced. In this case it is defined by

$$\Phi_\pi(g) = \pi(a(g)), \qquad g \in G,$$

where a is the Lie homomorphism from SO(4) to SO(3) given in (1). It is clear that Φ_π is an irreducible representation of SO(4) and hence a spherical function of type π (see Definition 2.1).

3. The differential operators D and E

To determine all irreducible spherical functions on G of type $\pi \in \hat{K}$, is equivalent to determine all analytic functions $\Phi : G \longrightarrow \mathrm{End}(V_\pi)$ such that

i) $\Phi(k_1 g k_2) = \pi(k_1)\Phi(g)\pi(k_2)$, for all $k_1, k_2 \in K$, $g \in G$, and $\Phi(e) = I$.
ii) $[\Delta_1 \Phi](g) = \widetilde{\lambda}\Phi(g)$, $[\Delta_2 \Phi](g) = \widetilde{\mu}\Phi(g)$ for all $g \in G$ and for some $\widetilde{\lambda}, \widetilde{\mu} \in \mathbb{C}$.

Instead of looking at an irreducible spherical function Φ of type π, we use the auxiliary function Φ_π to look at the function $H : G \longrightarrow \operatorname{End}(V_\pi)$ defined by

$$H(g) = \Phi(g)\Phi_\pi(g)^{-1}. \tag{7}$$

Observe that H is well defined on G because Φ_π is a representation of G. This function H, associated to the spherical function Φ, satisfies

i) $H(e) = I$.
ii) $H(gk) = H(g)$, for all $g \in G, k \in K$.
iii) $H(kg) = \pi(k)H(g)\pi(k^{-1})$, for all $g \in G, k \in K$.

The fact that Φ is an eigenfunction of Δ_1 and Δ_2 makes the function H into an eigenfunction of certain differential operators D and E on G to be determined now. Let us define

$$D(H) = Y_4^2(H) + Y_5^2(H) + Y_6^2(H), \tag{8}$$
$$E(H) = \big(- Y_4(H)Y_3(\Phi_\pi) + Y_5(H)Y_2(\Phi_\pi) - Y_6(H)Y_1(\Phi_\pi)\big)\Phi_\pi^{-1}. \tag{9}$$

Proposition 3.1. *For any $H \in C^\infty(G) \otimes \operatorname{End}(V_\pi)$ right invariant under K, the function $\Phi = H\Phi_\pi$ satisfies $\Delta_1\Phi = \widetilde{\lambda}\Phi$ and $\Delta_2\Phi = \widetilde{\mu}\Phi$ if and only if H satisfies $DH = \lambda H$ and $EH = \mu H$, with*

$$\lambda = -4\widetilde{\lambda}, \quad \mu = -\tfrac{1}{4}\ell(\ell+2) + \widetilde{\mu} - \widetilde{\lambda}.$$

Proof. First we observe that $Z_4(\Phi_\pi) = Z_5(\Phi_\pi) = Z_6(\Phi_\pi) = 0$, because Φ_π is a representation of G and $\dot{a}(Z_j) = 0$ for $j = 4, 5, 6$. In fact

$$Z_j(\Phi_\pi)(g) = \tfrac{d}{dt}\Big|_{t=0} [\Phi_\pi(g)\Phi_\pi(\exp tZ_j)] = \Phi_\pi(g)\dot{\pi}(\dot{a}(Z_j)) = 0.$$

On the other hand, since H is right invariant under K, we have that $Y_1(H) = Y_2(H) = Y_3(H) = 0$. Since $[Y_3, Y_4] = 0$, $[Y_2, Y_5] = 0$ and $[Y_1, Y_6] = 0$ we have that $Z_j^2(H) = \frac{1}{4}Y_j^2(H)$, for $j = 4, 5, 6$. Therefore we obtain

$$\Delta_1(H\Phi_\pi) = -\sum_{j=4}^{6} Z_j^2(H)\,\Phi_\pi = -\frac{1}{4}\sum_{j=4}^{6} Y_j^2(H)\,\Phi_\pi = -\frac{1}{4}D(H)\Phi_\pi.$$

On the other hand we have

$$\Delta_2(H\Phi_\pi) = -\sum_{j=1}^{3} \big(Z_j^2(H)\,\Phi_\pi + 2Z_j(H)Z_j(\Phi_\pi) + HZ_j^2(\,\Phi_\pi)\big),$$

Observe that $Z_1(H) = \frac{1}{2}Y_4(H)$. Since $Z_1 = Y_3 - Z_4$ we have $Z_1(\Phi_\pi) = Y_3(\Phi_\pi)$ and $Z_1^2(\Phi_\pi) = Y_3^2(\Phi_\pi)$. Similar results holds for Z_2 and Z_3. Therefore

$$\begin{aligned}\Delta_2(H\Phi_\pi) &= -\big(Z_1^2(H)\,\Phi_\pi + Y_4(H)Y_3(\Phi_\pi) + HY_3^2(\,\Phi_\pi)\big)\\ &\quad - \big(Z_2^2(H)\,\Phi_\pi - Y_5(H)Y_2(\Phi_\pi) + HY_2^2(\,\Phi_\pi)\big)\\ &\quad - \big(Z_3^2(H)\,\Phi_\pi + Y_6(H)Y_1(\Phi_\pi) + HY_1^2(\,\Phi_\pi)\big)\\ &= -\frac{1}{4}D(H)\,\Phi_\pi + E(H)\,\Phi_\pi + H\Delta_K(\Phi_\pi)\\ &= -\frac{1}{4}D(H)\,\Phi_\pi + E(H)\,\Phi_\pi + H\Phi_\pi\dot{\pi}(\Delta_K).\end{aligned}$$

Since $\Delta_K \in D(G)^K$ Schur's Lemma tells us that $\dot\pi(\Delta_K) = cI$. Now we have $\Delta_1(H\Phi_\pi) = \widetilde{\lambda} H\Phi_\pi$ and $\Delta_2(H\Phi_\pi) = \widetilde{\mu} H\Phi_\pi$ if and only if $D(H) = \lambda H$ and $E(H) = \mu H$, where

$$\widetilde{\lambda} = -\frac{1}{4}\lambda \qquad \text{and} \qquad \widetilde{\mu} = c + \widetilde{\lambda} + \mu.$$

To compute the constant c we take a highest weight vector $v \in V_\pi$, and write Y_1, Y_2, Y_3 in terms of the basis $\{e, f, g\}$ introduced in (3). It follows that

$$\begin{aligned}
\dot\pi(\Delta_K)v &= \dot\pi\left(-\left(\tfrac{-i}{2}(e+f)\right)^2 - \left(\tfrac{-1}{2}(e-f)\right)^2 - \left(\tfrac{i}{2}h\right)^2\right)v \\
&= \frac{-1}{4}\dot\pi\left(-2ef - 2fe - h^2\right)v = \frac{1}{4}\dot\pi\left(2(fe+h) + 2fe + h^2\right)v \\
&= \frac{1}{4}\left(2\ell + \ell^2\right)v = \frac{\ell(\ell+2)}{4}\,v\,.
\end{aligned}$$

Thus $c = \ell(\ell+2)/4$ completing the proof of the proposition. □

3.1. **Reduction to G/K.**

The quotient G/K is the sphere S^3, moreover the canonical diffeomorphism is given by $gK \mapsto (g_{14}, g_{24}, g_{34}, g_{44}) \in S^3$.

The function H associated to the spherical function Φ is right invariant under K, then it may be considered as a function on S^3, that we also called H. The differential operators D and E introduced in (8) and (9), define differential operators on S^3.

Lemma 3.2. *The differential operators D and E on G, define differential operators D and E acting on $C^\infty(S^3) \otimes \operatorname{End}(V_\pi)$.*

Proof. The only thing we need to prove is that D and E preserve the subspace $C^\infty(G)^K \otimes \operatorname{End}(V_\pi)$.

First we prove that D commutes with Y_1. Since $[Y_1, Y_4] = -Y_5$, $[Y_1, Y_5] = Y_4$ and $[Y_1, Y_6] = 0$ we have

$$\begin{aligned}
Y_1 D &= Y_1Y_4^2 + Y_1Y_5^2 + Y_1Y_6^2 = -Y_5Y_4 + Y_4Y_1Y_4 + Y_4Y_5 + Y_5Y_1Y_5 + Y_1Y_6^2 \\
&= -Y_5Y_4 - Y_4Y_5 + {Y_4}^2Y_1 + Y_4Y_5 + Y_5Y_4 + {Y_5}^2Y_1 + Y_6^2Y_1 \\
&= {Y_4}^2Y_1 + {Y_5}^2Y_1 + Y_6^2Y_1 = DY_1.
\end{aligned}$$

Similarly we get that $Y_2D = DY_2$, therefore D commutes with every $X \in \mathfrak{k}$, then D preserves $C^\infty(G)^K \otimes \operatorname{End}(V_\pi)$, since K is connected.

Let us check that E has the the same property. For that it is enough to verify that for any $X \in \mathfrak{k}$ and all $H \in C^\infty(G)^K \otimes \operatorname{End}(V_\pi)$ we have $X(EH) = 0$. Now we use that $\Phi_\pi\dot\pi(X) = X(\Phi_\pi)$ for any $X \in \mathfrak{k}$ and we get that $X(EH) = 0$ is equivalent to

$$\begin{aligned}
&- XY_4(H)\Phi_\pi\dot\pi(Y_3) - Y_4(H)\Phi_\pi\dot\pi(X)\dot\pi(Y_3) + Y_4(H)\Phi_\pi\dot\pi(Y_3)\dot\pi(X) \\
&+ XY_5(H)\Phi_\pi\dot\pi(Y_2) + Y_5(H)\Phi_\pi\dot\pi(X)\dot\pi(Y_2) - Y_5(H)\Phi_\pi\dot\pi(Y_2)\dot\pi(X) \\
&- XY_6(H)\Phi_\pi\dot\pi(Y_1) - Y_6(H)\Phi_\pi\dot\pi(X)\dot\pi(Y_1) + Y_6(H)\Phi_\pi\dot\pi(Y_1)\dot\pi(X) = 0.
\end{aligned}$$

Again we take $X = Y_1$, having then

$$\begin{aligned}
&- [Y_1, Y_4](H)\Phi_\pi\dot\pi(Y_3) - Y_4(H)\Phi_\pi\dot\pi([Y_1, Y_3]) \\
&+ [Y_1, Y_5](H)\Phi_\pi\dot\pi(Y_2) + Y_5(H)\Phi_\pi\dot\pi([Y_1, Y_2]) = 0,
\end{aligned}$$

since $XH = 0$ for any $X \in \mathfrak{k}$. But that is equivalent to

$$Y_5(H)\Phi_\pi\dot{\pi}(Y_3) - Y_4(H)\Phi_\pi\dot{\pi}(Y_2) + Y_4(H)\Phi_\pi\dot{\pi}(Y_2) - Y_5(H)\Phi_\pi\dot{\pi}(Y_3) = 0,$$

which is true. Similarly we obtain that $Y_2(EH) = 0$ and therefore we reached the result, since $\{Y_1, Y_2\}$ generates $\mathfrak{k}$. $\square$

Now we give the expressions of the operators D and E in the coordinate system $p : (S^3)^+ \longrightarrow \mathbb{R}^3$ given by

$$p(x) = \left(\frac{x_1}{x_4}, \frac{x_2}{x_4}, \frac{x_3}{x_4}\right) = (y_1, y_2, y_3).$$

The proofs of the following propositions require simple but lengthly computations and we don't include them in this paper, they can be found in [Z].

Proposition 3.3. *For any $H \in C^\infty(\mathbb{R}^3) \otimes \operatorname{End}(V_\pi)$ we have*

$$\begin{aligned} D(H)(y) = (1 + \|y\|^2)\Big(&(y_1^2+1)H_{y_1y_1} + (y_2^2+1)H_{y_2y_2} + (y_3^2+1)H_{y_3y_3} \\ &+ 2(y_1y_2H_{y_1y_2} + y_2y_3H_{y_2y_3} + y_1y_3H_{y_1y_3}) + 2(y_1H_{y_1} + y_2H_{y_2} + y_3H_{y_3})\Big). \end{aligned}$$

Proposition 3.4. *For any $H \in C^\infty(\mathbb{R}^3) \otimes \operatorname{End}(V_\pi)$ we have*

$$\begin{aligned} E(H)(y) = {} & H_{y_1}\dot{\pi}\begin{pmatrix} 0 & -y_2-y_1y_3 & -y_3+y_1y_2 \\ y_2+y_1y_3 & 0 & -1-y_1^2 \\ y_3-y_1y_2 & 1+y_1^2 & 0 \end{pmatrix} \\ & + H_{y_2}\dot{\pi}\begin{pmatrix} 0 & -y_2y_3+y_1 & 1+y_2^2 \\ y_2y_3-y_1 & 0 & -y_3-y_1y_2 \\ -1-y_2^2 & y_3+y_1y_2 & 0 \end{pmatrix} + H_{y_3}\dot{\pi}\begin{pmatrix} 0 & -1-y_3^2 & y_1+y_2y_3 \\ 1+y_3^2 & 0 & y_2-y_1y_3 \\ -y_1-y_2y_3 & -y_2+y_1y_3 & 0 \end{pmatrix}. \end{aligned}$$

3.2. **Reduction to one variable.**

We are interested in considering the differential operators D and E given in Propositions 3.3 and 3.4 applied to functions $H \in C^\infty(\mathbb{R}^3) \otimes \operatorname{End}(V_\pi)$ such that

$$H(ky) = \pi(k)H(y)\pi(k)^{-1}, \qquad \text{for all } k \in K, y \in \mathbb{R}^3.$$

By taking into account the K-orbit structure of $\mathbb{R}^3$, this property of H allows us to find ordinary differential operators $\widetilde{D}$ and $\widetilde{E}$ defined on the interval $(0, \infty)$ such that

$$(D\,H)(r, 0, 0) = (\widetilde{D}\widetilde{H})(r), \qquad (E\,H)(r, 0, 0) = (\widetilde{E}\widetilde{H})(r),$$

where $\widetilde{H}(r) = H(r, 0, 0)$. Recall that the interval $[0, \infty)$ parameterizes the set of K-orbits in $\mathbb{R}^3$.

In order to give the explicit expressions of the differential operators $\widetilde{D}$ and $\widetilde{E}$, and starting from Propositions 3.3 and 3.4, we need to compute a number of second order partial derivatives of the function $H : \mathbb{R}^3 \longrightarrow \operatorname{End}(V_\pi)$ at the points $(r, 0, 0)$, with $r > 0$. Given $y = (y_1, y_2, y_3) \in \mathbb{R}^3$ in a neighborhood of $(r, 0, 0)$, $r > 0$, we need a smooth function onto $K = \mathrm{SO}(3)$ such that carries the point y to the meridian $\{(r, 0, 0) : r > 0\}$. A good choice is the following function

$$A(y) = \frac{1}{\|y\|}\begin{pmatrix} y_1 & -y_2 & -y_3 \\ y_2 & \frac{-y_2^2}{\|y\|+y_1} + \|y\| & \frac{-y_2y_3}{\|y\|+y_1} \\ y_3 & \frac{-y_2y_3}{\|y\|+y_1} & \frac{-y_3^2}{\|y\|+y_1} + \|y\| \end{pmatrix}. \tag{10}$$

Then

$$y = A(y)(\|y\|, 0, 0)^t.$$

It is easy to verify that $A(y)$ is a matrix in SO(3) and it is well defined in $\mathbb{R}^3 - \{\,(y_1, 0, 0) \in \mathbb{R}^3 : y_1 \le 0\,\}$.

The proofs of the following propositions are similar to those in the case of the complex projective plane considered in [GPT2]. Let us consider the following elements in $\mathfrak{k}$

$$A_1 = E_{21} - E_{12}, \qquad A_2 = E_{31} - E_{13}, \qquad A_3 = E_{32} - E_{23}. \tag{11}$$

Proposition 3.5. *For $r > 0$ we have*

$$\begin{aligned}
\frac{\partial H}{\partial y_1}(r,0,0) =& \frac{d\widetilde{H}}{dr}(r),\\
\frac{\partial H}{\partial y_2}(r,0,0) =& \frac{1}{r}\left[\dot{\pi}\left(A_1\right), \widetilde{H}(r)\right], \qquad \frac{\partial H}{\partial y_3}(r,0,0) = \frac{1}{r}\left[\dot{\pi}\left(A_2\right), \widetilde{H}(r)\right].
\end{aligned}$$

Proposition 3.6. *For $r > 0$ we have*

$$\begin{aligned}
\frac{\partial^2 H}{\partial {y_1}^2}(r,0,0) =& \frac{d^2\widetilde{H}}{dr^2}(r),\\
\frac{\partial^2 H}{\partial {y_2}^2}(r,0,0) =& \frac{1}{r^2}\left(r\frac{d\widetilde{H}}{dr} + \dot{\pi}\left(A_1\right)^2 \widetilde{H}(r) + \widetilde{H}(r)\dot{\pi}\left(A_1\right)^2 - 2\dot{\pi}\left(A_1\right)\widetilde{H}(r)\dot{\pi}\left(A_1\right) \right),\\
\frac{\partial^2 H}{\partial {y_3}^2}(r,0,0) =& \frac{1}{r^2}\left(r\frac{d\widetilde{H}}{dr} + \dot{\pi}\left(A_2\right)^2 \widetilde{H}(r) + \widetilde{H}(r)\dot{\pi}\left(A_2\right)^2 - 2\dot{\pi}\left(A_2\right)\widetilde{H}(r)\dot{\pi}\left(A_2\right) \right).
\end{aligned}$$

Now we can obtain the explicit expressions of the differential operators $\tilde{D}$ and $\tilde{E}$.

Theorem 3.7. *For $r > 0$ we have*

$$\begin{aligned}
\widetilde{D}(\widetilde{H})(r) = (1+r^2)^2\frac{d^2\widetilde{H}}{dr^2} &+ 2\frac{(1+r^2)^2}{r}\frac{d\widetilde{H}}{dr}\\
&+ \frac{(1+r^2)}{r^2}\left(\dot{\pi}(A_1)^2\,\widetilde{H}(r) + \widetilde{H}(r)\dot{\pi}(A_1)^2 - 2\dot{\pi}(A_1)\widetilde{H}(r)\dot{\pi}(A_1)\right)\\
&+ \frac{(1+r^2)}{r^2}\left(\dot{\pi}(A_2)^2\,\widetilde{H}(r) + \widetilde{H}(r)\dot{\pi}(A_2)^2 - 2\dot{\pi}(A_2)\widetilde{H}(r)\dot{\pi}(A_2)\right).
\end{aligned}$$

Proof. Since $\widetilde{D}(\widetilde{H})(r) = D(H)(r,0,0)$, from Proposition 3.3 we have

$$\widetilde{D}(\widetilde{H})(r) = (1+r^2)\Big((1+r^2)H_{y_1y_1} + H_{y_2y_2} + H_{y_3y_3} + 2rH_{y_1}\Big).$$

Using Propositions 3.6 and 3.6 we get

$$\begin{aligned}
\widetilde{D}(\widetilde{H})(r) =& (1+r^2)\Bigg[(1+r^2)\frac{d^2\widetilde{H}}{dr^2}(r) + 2r\frac{d\widetilde{H}}{dr}(r) + \frac{2}{r}\frac{d\widetilde{H}}{dr}\\
&+ \frac{1}{r^2}\left(\dot{\pi}\left(A_1\right)^2 \widetilde{H}(r) + \widetilde{H}(r)\dot{\pi}\left(A_1\right)^2 - 2\dot{\pi}\left(A_1\right)\widetilde{H}(r)\dot{\pi}\left(A_1\right)\right)\\
&+ \frac{1}{r^2}\left(\dot{\pi}\left(A_2\right)^2 \widetilde{H}(r) + \widetilde{H}(r)\dot{\pi}\left(A_2\right)^2 - 2\dot{\pi}\left(A_2\right)\widetilde{H}(r)\dot{\pi}\left(A_2\right)\right)\Bigg].
\end{aligned}$$

Now the theorem follows easily.

□

Theorem 3.8. *For $r>0$ we have*
$$\begin{aligned}\widetilde{E}(\widetilde{H})(r) = &\frac{d\widetilde{H}}{dr}(1+r^2)\dot{\pi}(A_3) - \frac{1}{r}\left[\dot{\pi}(A_1), \widetilde{H}(r)\right]\dot{\pi}\left(rA_1+A_2\right)\\ &+\frac{1}{r}\left[\dot{\pi}(A_2), \widetilde{H}(r)\right]\dot{\pi}(A_1-rA_2).\end{aligned}$$

Proof. Since $\widetilde{E}(\widetilde{H})(r) = E(H)(r,0,0)$, from Proposition 3.4 we have

$$\widetilde{E}(\widetilde{H})(r) = H_{y_1}\dot{\pi}\begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & -1-r^2\\ 0 & 1+r^2 & 0\end{pmatrix} + H_{y_2}\dot{\pi}\begin{pmatrix} 0 & r & 1\\ -r & 0 & 0\\ -1 & 0 & 0\end{pmatrix} + H_{y_3}\dot{\pi}\begin{pmatrix} 0 & -1 & r\\ 1 & 0 & 0\\ -r & 0 & 0\end{pmatrix}.$$

Now, from Proposition 3.5 we get

$$\begin{aligned}\widetilde{E}(H)(r) = &\frac{d\widetilde{H}}{dr}(1+r^2)\dot{\pi}\left(A_3\right) - \frac{1}{r}\left[\dot{\pi}\left(A_1\right), \widetilde{H}(r)\right]\dot{\pi}\left(rA_1+A_2\right)\\ &+\frac{1}{r}\left[\dot{\pi}\left(A_2\right), \widetilde{H}(r)\right]\dot{\pi}\left(A_1-rA_2\right),\end{aligned}$$

which is the statement of the theorem. □

Theorems 3.7 and 3.8 are given in terms of linear transformations. Now we will give the corresponding statements in terms of matrices by choosing an appropriate basis of V_π. We take the $\mathfrak{sl}(2)$-triple $\{e,f,h\}$ in $\mathfrak{k}_{\mathbb{C}} \simeq \mathfrak{sl}(2,\mathbb{C})$ introduced in (3). If $\pi=\pi_\ell$ is the only irreducible representation of SO(3) with highest weight $\ell/2$, we recalled in Subsection 2.4 that there exists a basis $\mathcal{B}=\{v_j\}_{j=0}^{\ell}$ of V_π such that

$$\begin{aligned}&\dot{\pi}(h)v_j = (\ell-2j)v_j,\\ &\dot{\pi}(e)v_j = (\ell-j+1)v_{j-1}, \quad (v_{-1}=0),\\ &\dot{\pi}(f)v_j = (j+1)v_{j+1}, \quad (v_{\ell+1}=0).\end{aligned} \tag{12}$$

Proposition 3.9. *The function $\widetilde{H}$ associated to an irreducible spherical function Φ of type $\pi\in\hat{K}$ diagonalizes simultaneously in the basis $\mathcal{B}=\{v_j\}_{j=0}^{\ell}$ of V_π.*

Proof. Let us consider the subgroup $M=\{\, m_\theta : \theta\in\mathbb{R}\,\}$ of K, where
$$m_\theta = \begin{pmatrix} 1 & 0 & 0\\ 0 & \cos\theta & \sin\theta\\ 0 & -\sin\theta & \cos\theta\end{pmatrix}.$$

Then M is isomorphic to SO(2) and it fixes the points $(r,0,0)$ in $\mathbb{R}^3$. Also, since the function H satisfies $H(kg)=\pi(k)H(g)\,\pi(k^{-1})$ for all $k\in K$, we have that
$$\begin{aligned}\widetilde{H}(r) &= H(r,0,0) = H(m_\theta(r,0,0)^t) = \pi(m_\theta)H(r,0,0)\pi(m_\theta^{-1})\\ &= \pi(m_\theta)\widetilde{H}(r)\pi(m_\theta^{-1}).\end{aligned}$$

Hence, $\widetilde{H}(r)$ and $\pi(m_\theta)$ commute for every r in $\mathbb{R}$ and every m_θ in M.

On the other hand, notice that $m_\theta=\exp(\theta\frac{i}{2}h)$ and then $\pi(m_\theta)=\exp(\dot{\pi}(\theta\frac{i}{2}h))$, but from (12) we know that $\dot{\pi}(h)$ diagonalizes and that its eigenvalues have multiplicity one. Therefore the function $\widetilde{H}(r)$ diagonalizes simultaneously in the basis $\mathcal{B}=\{v_j\}_{j=0}^{\ell}$ of V_π.

□

Now we introduce the coordinate functions $\widetilde{h}_j(r)$ by means of

$$\widetilde{H}(r)v_j = \widetilde{h}_j(r)v_j, \tag{13}$$

and we identify $\widetilde{H}$ with the column vector

$$\widetilde{H}(r) = (\widetilde{h}_0(r), \dots, \widetilde{h}_\ell(r))^t. \tag{14}$$

Corollary 3.10. *The functions $\widetilde{H}(r)$, $(0 < r < \infty)$, satisfy $(\widetilde{D}\widetilde{H})(r) = \lambda\widetilde{H}(r)$ if and only if*

$$\begin{aligned}&(1+r^2)^2\widetilde{h}_j'' + 2\tfrac{(1+r^2)^2}{r}\widetilde{h}_j' + \tfrac{1+r^2}{r^2}(j+1)(\ell-j)(\widetilde{h}_{j+1}-\widetilde{h}_j)\\ &\quad + \tfrac{1+r^2}{r^2}j(\ell-j+1)(\widetilde{h}_{j-1}-\widetilde{h}_j) = \lambda\widetilde{h}_j,\end{aligned}$$

for all $j = 0, \dots, \ell$.

Proof. Using the basis $\mathcal{B} = \{v_j\}_{j=0}^{\ell}$ of V_π (see (12)) and writing the matrices A_1 and A_2 in terms of the $\mathfrak{sl}(2)$-triple $\{e,\ f,\ h\}$, see (3),

$$A_1 = E_{21} - E_{12} = \frac{i}{2}(e+f)\ , \qquad A_2 = E_{31} - E_{13} = \frac{1}{2}(e-f),$$

we have that Theorem 3.7 says that $(\widetilde{D}\widetilde{H})(r) = \lambda\widetilde{H}(r)$ if and only if

$$\begin{aligned}\lambda\widetilde{H}(r)\,v_j &= (1+r^2)^2\widetilde{H}''(r)\,v_j + 2\frac{(1+r^2)^2}{r}\widetilde{H}'(r)\,v_j\\ &\quad - \frac{(1+r^2)}{4r^2}\Big(\dot\pi(e+f)^2\,\widetilde{H}(r)\ + \widetilde{H}(r)\dot\pi(e+f)^2 - 2\dot\pi(e+f)\widetilde{H}(r)\dot\pi(e+f)\Big)\ v_j\\ &\quad + \frac{(1+r^2)}{4r^2}\Big(\dot\pi(e-f)^2\,\widetilde{H}(r)\ + \widetilde{H}(r)\dot\pi(e-f)^2 - 2\dot\pi(e-f)\widetilde{H}(r)\dot\pi(e-f)\Big)\ v_j,\end{aligned}$$

for $0 \le j \le \ell$.

As $[e,f] = h$ we have that this equivalent to

$$\begin{aligned}\lambda\widetilde{H}(r)v_j &= (1+r^2)^2\widetilde{H}''(r)\,v_j + 2\frac{(1+r^2)^2}{r}\widetilde{H}'(r)\,v_j\\ &\quad - \frac{(1+r^2)}{2r^2}\Big[(\dot\pi(h) + 2\dot\pi(f)\dot\pi(e)\,)\widetilde{H}(r)\,v_j\ + \widetilde{H}(r)(\dot\pi(h) + 2\dot\pi(f)\dot\pi(e))\,v_j\\ &\quad - 2(\dot\pi(e)\widetilde{H}(r)\dot\pi(f) + \dot\pi(f)\widetilde{H}(r)\dot\pi(e))\,v_j\Big],\end{aligned}$$

for $0 \le j \le \ell$. Now, using (12), we obtain that $(\widetilde{D}\widetilde{H})(r) = \lambda\widetilde{H}(r)$ if and only if

$$\begin{aligned}\lambda\widetilde{h}_j(r)\,v_j &= (1+r^2)^2\widetilde{h}_j''(r)\,v_j + 2\frac{(1+r^2)^2}{r}\widetilde{h}_j'(r)\,v_j\\ &\quad -\frac{(1+r^2)}{2r^2}\Big[((\ell-2j) + 2j(\ell-j+1)\,)\widetilde{h}_j(r)\,v_j\ + \widetilde{h}_j(r)((\ell-2j) + 2j(\ell-j+1))\,v_j\\ &\quad - 2((\ell-j)\widetilde{h}_{j+1}(r)(j+1) + j\widetilde{h}_{j-1}(r)(\ell-j+1))\,v_j\Big],\end{aligned}$$

for $0 \le j \le \ell$.

It can be easily checked that this is the required result. $\square$

Corollary 3.11. *The functions $\widetilde H(r)$, $(0<r<\infty)$, satisfy $(\widetilde E\widetilde H)(r)=\mu\widetilde H(r)$ if and only if*

$$\begin{aligned}
&-i(\ell-2j)\tfrac{1+r^2}{2}\widetilde h_j'+\tfrac{i}{2r}\Big((j+1)(\ell-j)(\widetilde h_{j+1}-\widetilde h_j)-j(\ell-j+1)(\widetilde h_{j-1}-\widetilde h_j)\Big)\\
&+\tfrac12\Big((j+1)(\ell-j)(\widetilde h_{j+1}-\widetilde h_j)+j(\ell-j+1)(\widetilde h_{j-1}-\widetilde h_j)\Big)=\mu\widetilde h_j,
\end{aligned}$$

for all $j=0,\dots,\ell$.

Proof. We proceed similarly to the proof of Corollary 3.10. Using the $\mathfrak{sl}(2)$-triple $\{e,\,f,\,h\}$ and the matrices A_1, A_2 and A_3 (see (11)), from Theorem 3.8 we have that $(\widetilde E\widetilde H)(r)=\mu\widetilde H(r)(r)$ if and only if

$$\begin{aligned}
\mu\widetilde H(r)\,v_j=&(1+r^2)H'(r)\dot\pi\left(A_3\right)\,v_j-\frac1r\left[\dot\pi\left(A_1\right),\widetilde H(r)\right]\dot\pi\left(rA_1+A_2\right)\,v_j\\
&+\frac1r\left[\dot\pi\left(A_2\right),\widetilde H(r)\right]\dot\pi\left(A_1-rA_2\right)\,v_j,
\end{aligned}$$

for every v_j in $\mathcal B=\{v_j\}_{j=0}^{\ell}$.

As in the proof of the Theorem 3.10, we write A_1, A_2 and A_3 in terms of $\{e,\,f,\,h\}$ (see (3)),

$$A_1=\frac i2(e+f)\,,\qquad A_2=\frac12(e-f),\qquad A_3=-\frac i2h.$$

Hence, $(\widetilde D\widetilde H)(r)=\lambda\widetilde H(r)$ if and only if

$$\begin{aligned}
\mu\widetilde H(r)v_j=&\frac1{4r}\left[\dot\pi\left(e+f\right),\widetilde H(r)\right]\dot\pi\left(r(e+f)-i(e-f)\right)v_j\\
&+\frac1{4r}\left[\dot\pi\left(e-f\right),\widetilde H(r)\right]\dot\pi\left(i(e+f)-r(e-f)\right)v_j-i\frac{1+r^2}{2}H'(r)\dot\pi\left(h\right)v_j,
\end{aligned}$$

for $0\le j\le\ell$. And that is equivalent to

$$\begin{aligned}
\mu\widetilde H(r)\,v_j=&-i\frac{1+r^2}{2}H'(r)\dot\pi\left(h\right)\,v_j+\frac1{2r}(r+i)\left[\dot\pi\left(e\right),\widetilde H(r)\right]\dot\pi(f)\,v_j\\
&+\frac1{2r}(r-i)\left[\dot\pi\left(f\right),\widetilde H(r)\right]\dot\pi(e)\,v_j,
\end{aligned}$$

for $0\le j\le\ell$.

Finally, we use (12) to obtain

$$\begin{aligned}
\mu\widetilde h_j\,v_j=&-i\frac{1+r^2}{2}\widetilde h_j'(2\ell-j)\,v_j+\frac1{2r}(r+i)(\ell-j)(\widetilde h_{j+1}-\widetilde h_j)(j+1)\,v_j\\
&+\frac1{2r}(r-i)j(\widetilde h_{j-1}-\widetilde h_j)(\ell-j+1)\,v_j,
\end{aligned}$$

for $0\le j\le\ell$. Therefore the corollary is proved. $\square$

In matrix notation, the differential operators $\widetilde D$ and $\widetilde E$ are given by

$$\widetilde D\widetilde H=(1+r^2)^2\widetilde H''+2\frac{(1+r^2)^2}{r}\widetilde H'+\frac{(1+r^2)}{r^2}(C_1+C_0)\widetilde H,$$

$$\widetilde{E}\widetilde{H} = -i\frac{1+r^2}{2}A_0\widetilde{H}' + \frac{i}{2r}(C_1 - C_0)\widetilde{H} + \frac{1}{2}(C_1 + C_0)\widetilde{H}.$$

where the matrices are given by

$$\begin{aligned} A_0 &= \textstyle\sum_{j=0}^{\ell}(\ell - 2j)E_{j,j}, \\ C_0 &= \textstyle\sum_{j=1}^{\ell} j(\ell - j + 1)(E_{j,j-1} - E_{j,j}), \\ C_1 &= \textstyle\sum_{j=0}^{\ell-1}(j+1)(\ell - j)(E_{j,j+1} - E_{j,j}). \end{aligned} \tag{15}$$

We introduce the change of variables $u = \frac{1}{\sqrt{1+r^2}}$, $u \in (0,1]$, and we put

$$H(u) = \widetilde{H}\left(\tfrac{\sqrt{1-u^2}}{u}\right) \text{ and } h_j(u) = \widetilde{h}_j\left(\tfrac{\sqrt{1-u^2}}{u}\right). \tag{16}$$

Under this change of variables, the differential operators $\widetilde{D}$ and $\widetilde{E}$ are converted into two new differential operators D and E. We get the following expressions for them,

$$DH = (1-u^2)\frac{d^2H}{du^2} - 3u\frac{dH}{du} + \frac{1}{1-u^2}(C_0 + C_1)H, \tag{17}$$

$$EH = \frac{i}{2}\sqrt{1-u^2}A_0\frac{dH}{du} + \frac{i}{2}\frac{u}{\sqrt{1-u^2}}(C_1 - C_0)H + \frac{1}{2}(C_0 + C_1)H. \tag{18}$$

Remark 3.12. At this point there is a slight abuse of notation, since D and E were used earlier to denote operators on $\mathbb{R}^3$.

4. Eigenfunctions of D

We are interested in determining the functions $H : (0,1) \longrightarrow \mathbb{C}^{\ell+1}$ that are eigenfunctions of the differential operator

$$DH = (1-u^2)\frac{d^2H}{du^2} - 3u\frac{dH}{du} + \frac{1}{1-u^2}(C_0 + C_1)H,$$

$u \in (0,1)$.

It is well known that such eigenfunctions are analytic functions on the interval $(0,1)$ and that the dimension of the corresponding eigenspace is $2(\ell+1)$.

The equation $DH = \lambda H$ is a coupled system of $\ell+1$ second order differential equations in the components $(h_0, \dots, h_\ell)$ of H, because the $(\ell+1)\times(\ell+1)$ matrix C_0+C_1 is not a diagonal matrix. But fortunately the matrix C_0+C_1 is a symmetric one, thus diagonalizable. Now we quote the following result from [GPT3].

Proposition 4.1. *The matrix $C_0 + C_1$ is diagonalizable. Moreover the eigenvalues are $-k(k+1)$ for $0 \le k \le \ell$ and the corresponding eigenvectors are given by $u_k = (U_{0,k}, \dots, U_{\ell,k})$ where*

$$U_{j,k} = {}_3F_2\left({-k,\, -j,\, k+1 \atop 1,\, -\ell}; 1\right), \tag{19}$$

an instance of the Hahn orthogonal polynomials.

We will need the following technical lemma.

Lemma 4.2. Let $U = (U_{j,k})$ be the matrix defined by

$$U_{j,k} = {}_3F_2\left({-k,\, -j,\, k+1 \atop 1,\, -\ell} ; 1 \right), \tag{20}$$

and let A_0, C_0 and C_1 be the matrices introduced in (15).

Then

$$\begin{aligned} U^{-1}A_0U &= Q_0 + Q_1, \\ U^{-1}(C_1 + C_0)U &= -V_0, \\ U^{-1}(C_1 - C_0)U &= Q_1J - Q_0(J+1), \end{aligned} \tag{21}$$

where

$$V_0 = \sum_{j=0}^{\ell-1} j(j+1)E_{j,j}\,, \qquad J = \sum_{j=0}^{\ell} jE_{jj}\,,$$

$$Q_0 = \sum_{j=0}^{\ell-1} \tfrac{(j+1)(\ell+j+2)}{2j+3} E_{j,j+1}\,, \quad Q_1 = \sum_{j=1}^{\ell} \tfrac{j(\ell-j+1)}{2j-1} E_{j,j-1}\,.$$

Proof. Firstly we observe that $U^{-1}(C_1 + C_0)U = -V_0$ is a direct consequence of Proposition 4.1. To prove that $U^{-1}A_0U = Q_0 + Q_1$ is equivalent to verify that

$$A_0U = U(Q_0 + Q_1).$$

By taking a look at the entry (j,k) for $j,k = 0,\dots,\ell$, we obtain that

$$(\ell - 2j)U_{j,k} = U_{j,k-1}\frac{k(\ell+k+1)}{2k+1} + U_{j,k+1}\frac{(k+1)(\ell-k)}{2k+1}.$$

From [AAR] (see page 346, equation (c)) we have that the last equality is satisfied by the Hahn polynomials.

Now we prove that

$$U^{-1}(C_1 - C_0)U = -Q_0(J+1) + Q_1J. \tag{22}$$

By using that $(C_0 + C_1)U = -UV_0$, (see (21)) we have that (22) is equivalent to

$$-2C_0U = U(-Q_0(J+1) + Q_1J + V_0).$$

To prove this identity, as we did before, we consider the entry (j,k), then we need to prove

$$\begin{aligned} &-2(C_0)_{j,j}U_{j,k} - 2(C_0)_{j,j-1}U_{j-1,k} = \\ &\qquad -U_{j,k-1}(Q_0)_{k-1,k}(J+1)_{k,k} + U_{j,k+1}(Q_1)_{k+1,k}J_{k,k} + U_{j,k}(V_0)_{k,k}, \end{aligned}$$

or equivalently we have to verify that

$$\begin{aligned} &2j(\ell-j+1)U_{j,k} - 2j(\ell-j+1)U_{j-1,k} = \\ &\quad - U_{j,k-1}\frac{k(\ell+k+1)}{2k+1}(k+1) + U_{j,k+1}\frac{(k+1)(\ell-k)}{2k+1}k + k(k+1)U_{j,k}. \end{aligned} \tag{23}$$

We use the same relation for the Hahn polynomials as before (see [AAR], page 346, equation (c))

$$(2\ell - j)(2k+1)U_{j,k} - k(\ell+k+1)U_{j,k-1} = (k+1)(\ell-k)U_{j,k+1},$$

therefore (23) becomes

$$\big(2j(\ell-j+1+k)-k(k+1+\ell)\big)U_{j,k}-2j(\ell-j+1)U_{j-1,k} = \\ -k(\ell+k+1)U_{j,k-1},$$

which is satisfied by the Hahn polynomials, see [KMcG]. □

If we define $\check H(u)=U^{-1}H(u)$, we get that $DH=\lambda H$ is equivalent to

$$(1-u^2)\frac{d^2\check H}{du^2}-3u\frac{d\check H}{du}-\frac{1}{1-u^2}V_0\check H=\lambda \check H,$$

where $V_0=\sum_{j=0}^{\ell-1} j(j+1)E_{j,j}$.

In this way we obtain that $DH=\lambda H$ if and only if the j-th component $\check h_j(u)$ of $\check H(u)$, for $0\le j\le \ell$, satisfies

$$(1-u^2)\,\check h_j''(u)-3\,u\check h_j'(u)-j(j+1)\frac{1}{(1-u^2)}\check h_j(u)-\lambda\check h_j(u)=0. \tag{24}$$

If we write $\lambda=-n(n+2)$ with $n\in\mathbb{C}$, and $\check h_j(u)=(1-u^2)^{j/2}p_j(u)$. Then, for $0<j<\ell$, $p_j(u)$ satisfies

$$(1-u^2)p_j''(u)-(2j+3)\,up_j'(u)+(n-j)(n+j+2)p_j(u)=0. \tag{25}$$

Making a new change of variable, $s=(1-u)/2$, $s\in\big[0,\frac12\big)$, and defining $\tilde p_j(s)=p_j(u)$ we have

$$s(1-s)\tilde p_j''(s)+(j+\tfrac32-(2j+3)\,s)\tilde p_j'(s)+(n-j)(n+j+2)\tilde p_j=0, \tag{26}$$

for $0<j<\ell$. This is a hypergeometric equation of parameters

$$a=-n+j\,,\qquad b=n+j+2\,,\qquad c=j+\tfrac32.$$

Hence every solution $\tilde p_j(s)$ of (26) for $0<s<\frac12$ is a linear combination of

$${}_2F_1\left(\begin{smallmatrix}-n+j,\,n+j+2\\ j+3/2\end{smallmatrix};s\right)\qquad\text{and}\qquad s^{-j-1/2}\,{}_2F_1\left(\begin{smallmatrix}-n-1/2,\,n+3/2\\ -j+1/2\end{smallmatrix};s\right).$$

Therefore, for $0\le j\le\ell$, any solution $\check h_j(u)$ of (24), for $0<u<1$, is of the form

$$\begin{aligned}\check h_j(u)=&\,a_j(1-u^2)^{j/2}\,{}_2F_1\left(\begin{smallmatrix}-n+j,\,n+j+2\\ j+3/2\end{smallmatrix};\tfrac{1-u}{2}\right)\\ &+b_j\,(1-u^2)^{-(j+1)/2}{}_2F_1\left(\begin{smallmatrix}-n-1/2,\,n+3/2\\ -j+1/2\end{smallmatrix};\tfrac{1-u}{2}\right),\end{aligned} \tag{27}$$

for some $a_j,b_j\in\mathbb{C}$.

Therefore we have proved the following theorem.

Theorem 4.3. *Let $H(u)$ be an eigenfunction of D with eigenvalue $\lambda=-n(n+2)$, $n\in\mathbb{C}$. Then H is of the form*

$$H(u)=UT(u)P(u)+US(u)Q(u)$$

where U is the matrix defined in (20),

$$T(u)=\sum_{j=0}^{\ell}(1-u^2)^{j/2}E_{jj},\qquad S(u)=\sum_{j=0}^{\ell}(1-u^2)^{-(j+1)/2}E_{jj},$$

$P=(p_0,\dots,p_\ell)^t$ and $Q=(q_0,\dots,q_\ell)^t$ are the vector valued functions given by

$$p_j(u)=a_j\,{}_2F_1\left(\begin{smallmatrix}-n+j,n+j+2\\ j+3/2\end{smallmatrix};\tfrac{1-u}{2}\right),$$
$$q_j(u)=b_j\,{}_2F_1\left(\begin{smallmatrix}-n-1/2,n+3/2\\ -j+1/2\end{smallmatrix};\tfrac{1-u}{2}\right),$$

where a_j and b_j are arbitrary complex numbers for $j=0,1,\dots,\ell$.

Going back to our problem of determining all irreducible spherical functions Φ, we recall that $\Phi(e)=I$, then the associated function $H\in C^\infty(\mathbb{R}^3)\otimes\mathrm{End}(V_\pi)$ satisfy $H(0,0,0)=I$. In the variable $r\in\mathbb{R}$ we have that $\lim_{r\to0^+}\widetilde{H}(r)=I$. Therefore we are interested in those eigenfunctions of D such that

$$\lim_{u\to1^-}H(u)=(1,1,\dots,1)\in\mathbb{C}^{\ell+1}.$$

From Theorem 4.3 we observe that

$$\lim_{u\to1^-}P(u)=(a_0,a_1,\dots,a_\ell)\qquad\text{and}\qquad\lim_{u\to1^-}Q(u)=(b_0,b_1,\dots,b_\ell).$$

Moreover the matrix $T(u)$ has limit when $u\to1^-$, while $S(u)$ has not. Therefore an eigenfunction H of D has limit when $u\to1^-$ if and only if the limit of $Q(u)$ when $u\to1^-$ is $(0,\dots,0)$. In such a case we have that

$$\lim_{u\to1^-}H(u)=\lim_{u\to1^-}UT(u)P(u)=U\,(a_0,0,\dots,0)^t=a_0(1,\dots,1)^t.$$

In this way we have proved the following result.

Corollary 4.4. Let $H(u)$ be an eigenfunction of D with eigenvalue $\lambda=-n(n+2)$, $n\in\mathbb{C}$, such that $\lim_{u\to1^-}H(u)$ exists. Then H is of the form

$$H(u)=UT(u)P(u)$$

with U the matrix defined in (20), $T(u)=\displaystyle\sum_{j=0}^{\ell}(1-u^2)^{j/2}E_{jj}$, and $P=(p_0,\dots,p_\ell)^t$ is the vector valued function given by

$$p_j(u)=a_j\,{}_2F_1\left(\begin{smallmatrix}-n+j,n+j+2\\ j+3/2\end{smallmatrix};\tfrac{1-u}{2}\right)$$

where a_j are arbitrary complex numbers for $j=1,2,\dots,\ell$. Also we have that $\lim_{u\to1^-}H(u)=a_0(1,1,\dots,1)^t$, then if $H(u)$ is associated to an irreducible spherical function we have $a_0=1$.

5. Eigenfunctions of D and E

In this section we shall study the simultaneous solutions of $DH(u)=\lambda H(u)$ and $EH(u)=\mu H(u)$, $0<u<1$.

We introduce a matrix function $P(u)$, defined from $H(u)$ by

$$H(u)=U\,T(u)\,P(u)\tag{28}$$

where U is the matrix defined in (20) and $T(u)=\sum_{j=0}^{\ell}(1-u^2)^{j/2}E_{jj}$.

The fact that H is an eigenfunction of the differential operators D and E, makes P an eigenfunction of the differential operators

$$\bar{D}=(UT(u))^{-1}\,D\,(UT(u))\qquad\text{and}\qquad\bar{E}=(UT(u))^{-1}\,E\,(UT(u)),$$

with, respectively, the same eigenvalues λ and μ. The explicit expressions of $\bar{D}$ and $\bar{E}$ are given in the following theorem.

Theorem 5.1. *The operators $\bar{D}$ and $\bar{E}$ defined above, are given by*

$$\bar{D}P = (1-u^2)P'' - uCP' - VP,$$
$$\bar{E}P = \tfrac{i}{2}\left((1-u^2)Q_0 + Q_1\right)P' - \tfrac{i}{2}uMP - \tfrac{1}{2}V_0P,$$

where

$$\begin{aligned} C &= \sum_{j=0}^{\ell}(2j+3)E_{jj}, & V &= \sum_{j=0}^{\ell} j(j+2)E_{jj},\\ Q_0 &= \sum_{j=0}^{\ell-1} \tfrac{(j+1)(\ell+j+2)}{2j+3}E_{j,j+1}, & Q_1 &= \sum_{j=1}^{\ell} \tfrac{j(\ell-j+1)}{2j-1}E_{j,j-1},\\ M &= \sum_{j=0}^{\ell-1}(j+1)(\ell+j+2)E_{j,j+1}, & V_0 &= \sum_{j=0}^{\ell-1} j(j+1)E_{jj}. \end{aligned} \tag{29}$$

Proof. Let $H = H(u) = UT(u)P(u)$. We start by computing $D(H)$ for the differential operator D introduced in (17).

$$\begin{aligned} DH &= (1-u^2)UTP'' + \big(2(1-u^2)UT' - 3uUT\big)P'\\ &\quad + \big((1-u^2)UT'' - 3uUT' + \tfrac{1}{1-u^2}(C_0+C_1)UT\big)P\\ &= UT\Big((1-u^2)P'' + \big(2(1-u^2)T^{-1}T' - 3u\big)P'\\ &\qquad + \Big((1-u^2)T^{-1}T'' - 3uT^{-1}T' + \frac{1}{1-u^2}T^{-1}U^{-1}(C_0+C_1)UT\Big)P\Big). \end{aligned}$$

Since T is a diagonal matrix we easily compute

$$T^{-1}(u)T'(u) = -\tfrac{u}{(1-u^2)}\sum_{j=0}^{\ell} j\,E_{jj}, \quad T^{-1}T''(u) = \tfrac{1}{(1-u^2)^2}\sum_{j=0}^{\ell} j((j-1)u^2 - 1)\,E_{jj}.$$

Also from (21) we have that $U^{-1}(C_0+C_1)U = -V_0$. Since V_0 is a diagonal matrix it commutes with T and we get

$$\begin{aligned} &(1-u^2)T^{-1}T'' - 3uT^{-1}T' + \frac{1}{1-u^2}T^{-1}U^{-1}(C_0+C_1)UT\\ &\qquad = \frac{1}{(1-u^2)}\sum_{j=0}^{\ell}\big(j(j-1)u^2 - j + 3ju^2 - j(j+1)\big)E_{jj} = -V. \end{aligned}$$

Now for the differential operator E introduced in (18) we compute $E(H)$ with $H(u) = UT(u)P(u)$.

$$\begin{aligned} EH &= \frac{i}{2}\sqrt{1-u^2}A_0UTP'\\ &\quad + \Big(\frac{i}{2}\sqrt{1-u^2}A_0UT' + \frac{i}{2}\frac{u}{\sqrt{1-u^2}}(C_1-C_0)UT + \frac{1}{2}(C_0+C_1)UT\Big)P\\ &= UT\Big(\frac{i}{2}\sqrt{1-u^2}T^{-1}U^{-1}A_0UTP' + \Big(\frac{i}{2}\sqrt{1-u^2}T^{-1}U^{-1}A_0UT'\\ &\quad + \frac{i}{2}\frac{u}{\sqrt{1-u^2}}T^{-1}U^{-1}(C_1-C_0)UT + \frac{1}{2}T^{-1}U^{-1}(C_0+C_1)UT\Big)P\Big). \end{aligned}$$

From Lemma 4.2 above we have that $U^{-1}A_0U = Q_0 + Q_1$. By using that $T = \sum_{j=0}^{\ell}(1-u^2)^{j/2}E_{jj}$ we get

$$\sqrt{1-u^2}\,T^{-1}U^{-1}A_0UT = \sqrt{1-u^2}\,T^{-1}(Q_0+Q_1)T = (1-u^2)Q_0 + Q_1.$$

From (21) and the fact that T is diagonal we have that $T^{-1}U^{-1}(C_0+C_1)UT = -V_0$. Then it only remains to prove that

$$\sqrt{1-u^2}T^{-1}U^{-1}A_0UT' + \frac{u}{\sqrt{1-u^2}}T^{-1}U^{-1}(C_1-C_0)UT = -uM. \tag{30}$$

Since $T'(u) = \frac{-u}{1-u^2}JT(u)$, where $J = \sum_{j=0}^{\ell} jE_{jj}$, we have to prove that

$$T^{-1}\big(U^{-1}A_0UJ - U^{-1}(C_1-C_0)U\big)T = \sqrt{1-u^2}M. \tag{31}$$

From Lemma 4.2 we have

$$\begin{aligned} U^{-1}A_0UJ - U^{-1}(C_1-C_0)U &= (Q_1+Q_0)J - Q_1J + Q_0(J+1) = Q_0(2J+1) \\ &= \sum_{j=0}^{\ell-1}(j+1)(\ell+j+2)E_{j,j+1} = M. \end{aligned}$$

Since $T = \sum_{j=0}^{\ell}(1-u^2)^{j/2}E_{jj}$, (31) is satisfied and this completes the proof of the theorem. $\square$

The function P is an eigenfunction of the differential operator $\bar{D}$ if and only if the function $H = UT(u)P(u)$ is an eigenfunction of the differential operator D. From Theorem 4.3 we have the explicit expression of the function $P(u) = (p_0(u),\dots,p_\ell(u))^t$,

$$p_j(u) = a_j\,{}_2F_1\left(\begin{smallmatrix} -n+j,n+j+2 \\ j+3/2 \end{smallmatrix};\tfrac{1-u}{2}\right) + b_j\,(1-u^2)^{-(j+1/2)}{}_2F_1\left(\begin{smallmatrix} -n-1/2,n+3/2 \\ -j+1/2 \end{smallmatrix};\tfrac{1-u}{2}\right),$$

where a_j and b_j are in $\mathbb{C}$, for $0 \le j \le \ell$.

Since we are interested in determining the irreducible spherical functions of the pair (G,K), we need to study the simultaneous eigenfunctions of D and E such that there exists a finite limit of the function H when $u \to 1^-$.

From Theorem 4.3 we have that $\lim_{u\to1^-} H(u)$ is finite if and only if

$$\lim_{u\to1^-} b_j\,(1-u^2)^{-(j+1)/2}{}_2F_1\left(\begin{smallmatrix} -n-1/2,n+3/2 \\ -j+1/2 \end{smallmatrix};\tfrac{1-u}{2}\right), \qquad \text{for all } 0 \le j \le \ell,$$

exists and it is finite. This is true if and only if $b_j = 0$ for all $0 \le j \le \ell$. Therefore $\lim_{u\to1^-} H(u)$ is finite if and only if $\lim_{u\to1^-} P(u)$ is finite.

From Corollary 4.4 we know that an eigenfunction $P = P(u)$ of $\bar{D}$ in the interval $(0,1)$ has a finite limit as $u \to 1^-$ if and only if P is analytic at $u = 1$. Let us now consider the following vector space of functions into $\mathbb{C}^{\ell+1}$,

$$W_\lambda = \left\{\, P = P(u) \text{ analytic in } (0,1] : \bar{D}P = \lambda P \,\right\}.$$

A function $P \in W_\lambda$ is characterized by $P(1) = (a_0,\cdots,a_\ell)$. Thus the dimension of W_λ is $\ell+1$ and the isomorphism $W_\lambda \simeq \mathbb{C}^{\ell+1}$ is given by

$$\nu : W_\lambda \longrightarrow \mathbb{C}^{\ell+1}, \qquad P \mapsto P(1).$$

The differential operators D and E commute because they are closely related to the Casimir operators Δ_1 and Δ_2 which are in the center of the universal enveloping algebra $D(G)^G$. Then the differential operators $\bar{D}$ and $\bar{E}$ commute.

Proposition 5.2. *The linear space W_λ is stable under the differential operator $\bar{E}$ and it restricts to a linear map on W_λ. Moreover the following is a commutative diagram*

$$\begin{array}{ccc} W_\lambda & \xrightarrow{\bar{E}} & W_\lambda \\ \nu\big\downarrow & & \big\downarrow\nu \\ \mathbb{C}^{\ell+1} & \xrightarrow{L(\lambda)} & \mathbb{C}^{\ell+1} \end{array} \tag{32}$$

where $L(\lambda)$ is the $(\ell+1)\times(\ell+1)$ matrix

$$\begin{aligned} L(\lambda) &= -\tfrac{i}{2}Q_1C^{-1}(V+\lambda) - \tfrac{i}{2}M - \tfrac12 V_0 \\ &= -i\sum_{j=1}^{\ell} \tfrac{j(\ell-j+1)\big((j-1)(j+1)+\lambda\big)}{2(2j-1)(2j+1)} E_{j,j-1} - i\sum_{j=0}^{\ell-1}\tfrac{(j+1)(\ell+j+2)}{2}E_{j,j+1} \\ &\quad -\sum_{j=0}^{\ell}\tfrac{j(j+1)}{2}E_{jj}. \end{aligned}$$

Proof. The differential operator $\bar{E}$ takes analytic functions into analytic functions, because its coefficients are polynomials, see Theorem 5.1. A function $P\in W_\lambda$ is analytic, then $\lim_{u\to 1^-}\bar{E}P(u)$ is finite. On the other hand since $\bar{D}$ and $\bar{E}$ commute, the differential operator $\bar{E}$ preserves the eigenspaces of $\bar{D}$. This proves that W_λ is stable under $\bar{E}$. In particular $\bar{E}$ restricts to a linear map $L(\lambda)$ on W_λ, to be determined now.

From Theorem 5.1 we have

$$\nu(\bar{E}(P)) = (\bar{E}P)(1) = \tfrac{i}{2}Q_1P'(1) - \tfrac{i}{2}MP(1) - \tfrac12 V_0P(1).$$

But we can obtain $P'(1)$ in terms of $P(1)$. In fact, if we evaluate $\bar{D}P=\lambda P$ at $u=1$ then we get

$$P'(1) = -C^{-1}(V+\lambda)P(1).$$

Notice that C is an invertible matrix. Hence

$$\begin{aligned} \nu(\bar{E}(P)) &= -\tfrac{i}{2}Q_1C^{-1}(V+\lambda)P(1) - \tfrac{i}{2}MP(1) - \tfrac12 V_0P(1) \\ &= L(\lambda)P(1) = L(\lambda)\,\nu(P). \end{aligned}$$

This completes the proof of the proposition. □

Remark 5.3. If $\lambda=-n(n+2)$, with $n\in\mathbb{C}$ then we have

$$\begin{aligned} L(\lambda) &= i\sum_{j=1}^{\ell}\tfrac{j(\ell-j+1)(n-j+1)(n+j+1)}{2(2j-1)(2j+1)}E_{j,j-1} - i\sum_{j=0}^{\ell-1}\tfrac{(j+1)(\ell+j+2)}{2}E_{j,j+1} \\ &\quad -\sum_{j=0}^{\ell}\tfrac{j(j+1)}{2}E_{jj}. \end{aligned} \tag{33}$$

Corollary 5.4. *All eigenvalues μ of $L(\lambda)$ have geometric multiplicity one, that is all eigenspaces are one dimensional.*

Proof. A vector $a=(a_0,a_1,\dots,a_\ell)^t$ is an eigenvector of $L(\lambda)$ of eigenvalue μ, if and only if $\{a_j\}_{j=0}^{\ell}$ satisfies the following three term recursion relation for $j=0,\dots,\ell$,

$$i\,\tfrac{j(\ell-j+1)(n-j+1)(n+j+1)}{2(2j-1)(2j+1)}\;a_{j-1} - \tfrac{j(j+1)}{2}\;a_j - i\tfrac{(j+1)(\ell+j+2)}{2}\;a_{j+1} = \mu\,a_j, \tag{34}$$

where we interpret $a_{-1} = a_{\ell+1} = 0$. From these equations we see that the vector a is determined by a_0, which proves that the geometric multiplicity of the eigenvalue μ of $L(\lambda)$ is one. □

Finally we get the main result of this section which is the characterization of the simultaneous eigenfunctions H of the differential operators D and E in $(0, 1)$, which are continuous in $(0, 1]$. Recall that the irreducible spherical functions of the pair (G, K) give raise to such functions H.

Corollary 5.5. *Let $H(u)$ be a simultaneous eigenfunction of D and E in $(0, 1)$, continuous in $(0, 1]$, with respective eigenvalues $\lambda = -n(n + 2)$, $n \in \mathbb{C}$, and μ. Thus H is of the form*

$$H(u) = UT(u)P(u)$$

with U the matrix given in (20), $T(u) = \sum_{j=0}^{\ell}(1 - u^2)^{j/2}E_{jj}$, and $P = (p_0, \dots, p_\ell)^t$ is the vector valued function given by

$$p_j(u) = a_j\, {}_2F_1\left({-n+j, n+j+2 \atop j+3/2} ; \tfrac{1-u}{2} \right)$$

where $\{a_j\}_{j=0}^{\ell}$ satisfies the recursion relation (34).

Also we have that $H(1) = a_0(1, 1, \dots, 1)^t$, then if $H(u)$ is associated to an irreducible spherical function we have that $a_0 = 1$.

In S^3, the set

$$\left\{ x_\theta = (\sqrt{1 - \theta^2}, 0, 0, \theta) : \theta \in [-1, 1] \right\}$$

parameterizes all the K-orbits. Notice that for $\theta > 0$ we have that $x_\theta \in (S^3)^+$, and $p(x_\theta) = (\frac{\sqrt{1-\theta^2}}{\theta}, 0, 0)$. Therefore, in terms of the variable $r \in [0, \infty)$ we have that $r = \frac{\sqrt{1-\theta^2}}{\theta}$, and then in terms of the variable $u \in (0, 1]$ we get $u = \frac{1}{\sqrt{1+r^2}} = \theta$. Hence, given an irreducible spherical function Φ of type $\pi \in \hat{K}$, if we consider the associated function $H : S^3 \longrightarrow \mathrm{End}(V_\pi)$ defined by

$$H(g\,(0, 0, 0, 1)^t) = \Phi(g)\Phi_\pi^{-1}(g), \qquad g \in G,$$

we have that

$$H(\sqrt{1 - u^2}, 0, 0, u) = diag\{H(u)\} = diag\{UT(u)P(u)\}, \tag{35}$$

where $H(u)$, $u \in (0, 1]$, is the vector valued function given in Corollary 5.5 and $diag\{H(u)\}$ means the diagonal matrix valued function whose kk-entry is equal to the k-th entry of the vector valued function $H(u)$.

6. Eigenvalues of the spherical functions

The aim of this section is to use the representation theory of G to compute the eigenvalues of an irreducible spherical function Φ corresponding to the differential operators Δ_1 and Δ_2. From these eigenvalues we shall obtain the eigenvalues of the function H as eigenfunctions of D and E.

As we describe in Section 2 there exists a one to one correspondence between irreducible spherical functions of (G,K) of type $\delta \in \hat{K}$ and finite dimensional irreducible representations of G which contain the K-type δ. In fact every irreducible spherical function Φ of type $\delta \in \hat{K}$ is of the form

$$\Phi(g)v = P(\delta)\tau(g)v, \qquad g \in G, \qquad v \in P(\delta)V_\tau, \tag{36}$$

where (τ, V_τ) is a finite dimensional irreducible representation of G which contains the K-type δ and $P(\delta)$ is the projection of V_τ onto the K-isotypic component of type δ.

The irreducible finite dimensional representations τ of $G = \mathrm{SO}(4)$ are parameterized by a pair of integers (m_1, m_2) such that

$$m_1 \geq |m_2|,$$

while the irreducible finite dimensional representations π_ℓ of $K = \mathrm{SO}(3)$ are parameterized by $\ell \in 2\mathbb{Z}_{\geq}0$.

The representations $\tau_{(m_1,m_2)}$ restricted to SO(3), contains the representation π_ℓ if and only if

$$m_1 \geq \tfrac{\ell}{2} \geq |m_2|.$$

Therefore the equivalence classes of irreducible spherical functions of (G,K) of type π_ℓ are parameterized by the set of all pairs $(m_1, m_2) \in \mathbb{Z}^2$ such that

$$m_1 \geq \tfrac{\ell}{2} \geq |m_2|.$$

Theorem 6.1. *Let $\Phi_\ell^{(m_1,m_2)}$ be the spherical function of type π_ℓ, associated to the representation $\tau_{(m_1,m_2)}$ of G. Then*

$$\begin{aligned} \Delta_1 \Phi_\ell^{(m_1,m_2)} &= \tfrac{1}{4}(m_1 - m_2)(m_1 - m_2 + 2)\Phi_\ell^{(m_1,m_2)}, \\ \Delta_2 \Phi_\ell^{(m_1,m_2)} &= \tfrac{1}{4}(m_1 + m_2)(m_1 + m_2 + 2)\Phi_\ell^{(m_1,m_2)}. \end{aligned}$$

Proof. We start by observing that the eigenvalue of any irreducible spherical function Φ corresponding to a differential operator $\Delta \in D(G)^G$, given by $[\Delta\Phi](e)$, is a scalar multiple of the identity. Since Δ_1 and Δ_2 are in $D(G)^G$ we have that

$$[\Delta_1 \Phi_\ell^{(m_1,m_2)}](e) = \dot{\tau}_{(m_1,m_2)}(\Delta_1) \quad \text{and} \quad [\Delta_2 \Phi_\ell^{(m_1,m_2)}](e) = \dot{\tau}_{(m_1,m_2)}(\Delta_2).$$

These scalars can be computed by looking at the action of Δ_1 and Δ_2 on a highest weight vector v of the representation $\tau_{(m_1,m_2)}$, which highest weight is of the form $m_1\varepsilon_1 + m_2\varepsilon_2$.

Recall that

$$\begin{aligned} \Delta_1 &= (iZ_6)^2 + iZ_6 - (Z_5 + iZ_4)(Z_5 - iZ_4), \\ \Delta_2 &= (iZ_3)^2 + iZ_3 - (Z_2 + iZ_1)(Z_2 - iZ_1). \end{aligned}$$

Since $(Z_5 - iZ_4)$ and $(Z_2 - iZ_1)$ are positive roots vectors and $Z_6, Z_3 \in \mathfrak{h}_\mathbb{C}$, we get

$$\dot{\tau}_{(m_1,m_2)}(\Delta_1)v = \dot{\tau}_{(m_1,m_2)}(iZ_6)^2 v + \dot{\tau}_{(m_1,m_2)}(iZ_6)v = \tfrac{1}{4}(m_1 - m_2)(m_1 - m_2 + 2)v,$$
$$\dot{\tau}_{(m_1,m_2)}(\Delta_2)v = \dot{\tau}_{(m_1,m_2)}(iZ_3)^2 v + \dot{\tau}_{(m_1,m_2)}(iZ_3)v = \tfrac{1}{4}(m_1 + m_2)(m_1 + m_2 + 2)v.$$

This completes the proof of the theorem. □

Now we give the eigenvalues of the function H, associated to an irreducible spherical function, corresponding to the differential operators D and E.

Corollary 6.2. *The function H associated to the spherical function $\Phi_\ell^{(m_1,m_2)}$ satisfies $DH = \lambda H$ and $EH = \mu H$ with*

$$\begin{aligned} \lambda &= -(m_1 - m_2)(m_1 - m_2 + 2), \\ \mu &= -\frac{\ell(\ell+2)}{4} + (m_1+1)m_2. \end{aligned} \tag{37}$$

Proof. Let $\Phi = \Phi_\ell^{(m_1,m_2)}$. From Proposition 3.1 we have that $\Delta_1\Phi = \widetilde{\lambda}\Phi$ and $\Delta_2\Phi = \widetilde{\mu}\Phi$ if and only if $DH = \lambda H$ and $EH = \mu H$ where the relation between the eigenvalues of H and Φ is

$$\lambda = -4\widetilde{\lambda}, \qquad \mu = -\tfrac{1}{4}\ell(\ell+2) + \widetilde{\mu} - \widetilde{\lambda}.$$

Now the statement follows easily from Theorem 6.1. □

Remark 6.3. Notice that we have just proved that the eigenvalue λ of the operator D corresponding to a function H associated to an irreducible spherical function can be written in the form

$$\lambda = -n(n+2), \qquad \text{with} \quad n \in \mathbb{Z}_{\geq 0}. \tag{38}$$

Proposition 6.4. *If Φ is an irreducible spherical function of $\big(\mathrm{SO}(4), \mathrm{SO}(3)\big)$ then $\Phi(-e) = \pm I$. Moreover, if $\Phi = \Phi_\ell^{(m_1,m_2)}$ and $g \in \mathrm{SO}(4)$ then*

$$\Phi(-g) = \begin{cases} \Phi(g) & \text{if } m_1 + m_2 \equiv 0 \mod (2) \\ -\Phi(g) & \text{if } m_1 + m_2 \equiv 1 \mod (2). \end{cases}$$

Proof. As we mentioned in the Section 2, every irreducible spherical function of the pair $\big(\mathrm{SO}(4), \mathrm{SO}(3)\big)$ of type π, is of the form $\Phi(g) = P_\pi \tau(g)$, where $\tau \in \hat{\mathrm{SO}}(4)$ contains the K-type π and P_π is the projection onto the π-isotypic component of V_τ.

Let η be the highest weight of $\tau = (m_1, m_2) \in \hat{\mathrm{SO}}(4)$, i.e. $\eta = m_1 e_1 + m_2 e_2$ (see Subsection 2.4). We have that

$$-e = \exp\begin{pmatrix} 0 & \pi & 0 & 0 \\ -\pi & 0 & 0 & 0 \\ 0 & 0 & 0 & \pi \\ 0 & 0 & -\pi & 0 \end{pmatrix},$$

therefore if v is a highest weight vector in V_τ of weight η, we have

$$\tau(-e)v = e^{-\pi(m_1+m_2)i} v = \pm v.$$

Since $\tau(-e)$ commutes with $\tau(g)$ for all $g \in \mathrm{SO}(4)$, by Schur's Lemma $\tau(-e)$ is a multiple of the identity . Thus

$$\tau(-e) = \begin{cases} I, & \text{if } m_1 + m_2 \equiv 0 \mod (2) \\ -I, & \text{if } m_1 + m_2 \equiv 1 \mod (2) \end{cases}.$$

Therefore

$$\Phi(-g) = P_\pi \tau(-g) = P_\pi \tau(-e)\tau(g) = \tau(-e)\Phi(g).$$

Hence the proposition is proved. □

7. From P to Φ

7.1. **Correspondence between polynomials and spherical functions.**

In the previous sections we were interested in the irreducible spherical functions of a fixed K-type π and we arrived to the study of the space of vector valued eigenfunctions $P(u)$, analytic in $(0,1]$, of the differential operators $\bar{D}$ and $\bar{E}$,

$$\bar{D}P = (1-u^2)P'' - uCP' - VP,$$
$$\bar{E}P = \tfrac{i}{2}\left((1-u^2)Q_0 + Q_1\right)P' - \tfrac{i}{2}uMP - \tfrac{1}{2}V_0P,$$

see Theorem 5.1 and Corollary 5.5.

In this subsection we show that, given $\pi_\ell \in \hat{K}$, there is a one to one correspondence between the vector valued polynomial eigenfunctions $P(u)$ of $\bar{D}$ and $\bar{E}$ such that the first entry of the vector $P(1)$ is equal to 1, and the irreducible spherical functions Φ of type π_ℓ. For that we first need the following lemma.

Lemma 7.1. Given $\ell \in 2\mathbb{Z}_{\geq 0}$ and $n \in \mathbb{Z}_{\geq 0}$, for $0 \leq j \leq \ell$, the functions

$$p_j(u) = a_j \, {}_2F_1\left(\begin{smallmatrix} -n+j,\, n+j+2 \\ j+3/2 \end{smallmatrix}; \tfrac{1-u}{2}\right),$$

where $a_0 = 1$ and $\{a_j\}_{j=0}^{\ell}$ satisfies the recursion relation (34) for some μ, are polynomial functions. Furthermore, if $j \leq n$ then a_j is zero or the degree of p_j is equal to $n-j$, and if $n < j$ then $a_j = 0$. Also, when $n \leq \ell$ we have that a_n is not zero.

Proof. Let us consider the vector valued function $P(u) = (p_0(u), \dots, p_\ell(u))^t$. From Section 5 we know that $P(u)$ is a simultaneous eigenfunction of the operators $\bar{D}$ and $\bar{E}$ of the Theorem 5.1.

For $0 \leq j \leq n$, the hypergeometric function

$$ {}_2F_1\left(\begin{smallmatrix} -n+j,\, n+j+2 \\ j+3/2 \end{smallmatrix}; \tfrac{1-u}{2}\right) \tag{39}$$

is a multiple of the Gegenbauer polynomial $C_{n-j}^{j+1}(u)$ (see [AS], page 561). When $n < j \leq \ell$, we know that (39) is not a polynomial. But the expression of $\bar{E}$ in Theorem 5.1, which involves only three-diagonal matrices, tells us that for each j we have

$$\begin{aligned} \mu p_j = {}& \tfrac{i}{2}\left((1-u^2)\tfrac{(j+1)(\ell+j+2)}{2j+3}p'_{j+1} + \tfrac{j(\ell-j+1)}{2j-1}p'_{j-1}\right) \\ & - \tfrac{i}{2}u(j+1)(\ell+j+2)p_{j+1} - \tfrac{1}{2}j(j+1)p_j. \end{aligned} \tag{40}$$

Hence, if p_j and p_{j-1} are polynomials then

$$p(u) = \tfrac{(1-u^2)}{2j+3}p'_{j+1}(u) - up_{j+1}(u), \quad 0 \leq u \leq 1,$$

is also a polynomial function in u. Let $p_{j+1}(u) = \sum_{k\geq 0} b_k u^k$, therefore

$$p(u) = \sum_{k\geq 0} kb_k \tfrac{(1-u^2)}{2j+3}u^{k-1} - b_k u^{k+1}.$$

If we denote $b_{-2} = b_{-1} = 0$ we have

$$p(u) = \sum_{k\geq -2} \tfrac{k+2}{2j+3}b_{k+2}u^{k+1} - \tfrac{k}{2j+3}b_k u^{k+1} - b_k u^{k+1},$$

hence, for $k+1$ bigger than the degree of $p(u)$

$$b_{k+2} = b_k \frac{k+2j+3}{k+2},$$

then $|b_{k+2}| \geq |b_k|$ for $k+1$ sufficiently large. Therefore $b_k = 0$ for k bigger than the degree of $p(u)$. Thus p_{j+1} is a polynomial.

In other words, if p_{j-2} and p_{j-1} are polynomials, then so is p_j. Also recall that for $j > n$, (39) is an infinite series, but since $p_j(u) = a_j C_{n-j}^{j+1}(u)$ is a polynomial a_j is necessarily zero.

If a_n is zero we would have that every a_j is zero, by (34), but that is absurd because $a_0 = 1$. $\square$

Theorem 7.2. *Given $\ell \in 2\mathbb{Z}_{\geq 0}$, $n \in \mathbb{Z}_{\geq 0}$, every eigenfunction $P(u)$ of $\bar{D}$ and $\bar{E}$ analytic in $(0,1]$, with eigenvalues $\lambda = -n(n+2)$ and μ, respectively, corresponds up to a scalar to an irreducible spherical function Φ of type $\pi_\ell \in \hat{K}$.*

Proof. From Corollary 5.5 we know that $P(u) = (p_0(u), \ldots, p_\ell(u))^t$ is a simultaneous eigenfunction of $\bar{D}$ and $\bar{E}$, with eigenvalues $\lambda = -n(n+2)$ and μ respectively, if and only if for every $0 \leq j \leq \ell$

$$p_j(u) = a_j\, {}_2F_1\left({-n+j, n+j+2 \atop j+3/2} ; \tfrac{1-u}{2} \right),$$

where $\{a_j\}_{j=0}^{\ell}$ satisfies the recursion relation (34). Recall that $\{a_j\}_{j=0}^{\ell}$ satisfies the recursion relation (34) if and only if $a = (a_0, \ldots, a_\ell)^t$ is an eigenvector of the matrix L_λ with eigenvalue μ. From Corollary 5.4 we know that every eigenspace of L_λ is one dimensional, then up to scalars there are no more than $\ell+1$ eigenvectors. Therefore using Lemma 7.1 we conclude that up to scalars there are no more than $\min(\ell+1, n+1)$ eigenvectors of L_λ. In other words, given $\lambda = -n(n+2)$ with $n \in \mathbb{Z}_{\geq 0}$, up to scalars there are no more than $\min(\ell+1, n+1)$ simultaneous eigenfunctions $P(u)$ of $\bar{D}$ and $\bar{E}$ analytic in $(0,1]$, such that $\bar{D}P = \lambda P$.

Hence it is enough to prove that for each $\lambda = -n(n+2)$, $n \in \mathbb{Z}_{\geq}0$, there are exactly $\min(\ell+1, n+1)$ irreducible spherical functions of type $\pi_\ell \in K$ such that each one is associated to a different eigenfunction $P(u)$ of the operators $\bar{D}$ and $\bar{E}$ with $\bar{D}P = \lambda P$.

As we have said, the equivalence classes of irreducible spherical functions of (G,K) of type π_ℓ are parameterized by the set of all pairs $(m_1, m_2) \in \mathbb{Z}^2$ such that

$$m_1 \geq \tfrac{\ell}{2} \geq |m_2|.$$

To every irreducible spherical function $\Phi_\ell^{(m_1,m_2)}$ corresponds a vector valued eigenfunction $P_\ell^{(m_1,m_2)}$ of the operators $\bar{D}$ and $\bar{E}$ whose eigenvalues, according to Corollary 6.2, are respectively

$$\begin{aligned} \lambda_\ell^{(m_1,m_2)} &= -(m_1-m_2)(m_1-m_2+2), \\ \mu_\ell^{(m_1,m_2)} &= -\tfrac{\ell(\ell+2)}{4} + (m_1+1)m_2. \end{aligned}$$

Easily we see that for different pairs (m_1, m_2), such that $m_1 \geq \frac{\ell}{2} \geq |m_2|$, one has different eigenvalues $\lambda_\ell^{(m_1,m_2)}$ and $\mu_\ell^{(m_1,m_2)}$. Hence each irreducible spherical function $\Phi_\ell^{(m_1,m_2)}$ is associated to a different eigenfunction $P_\ell^{(m_1,m_2)}$.

If $m_1 - m_2 = n$, the function $P_\ell^{(m_1,m_2)}$ is an eigenfunction of the operators $\bar{D}$ and $\bar{E}$ with $\bar{D}P_\ell^{(m_1,m_2)} = -n(n+2)P_\ell^{(m_1,m_2)}$. Since there are exactly $\min(\ell+1, n+1)$

pairs (m_1, m_2) satisfying $m_1 \geq \frac{\ell}{2} \geq |m_2|$ and $m_1 - m_2 = n$, we have proved the corollary.

□

Recall from Corollary 5.5 that if $P(u) = (p_0(u), \dots, p_\ell(u))^t$ is a simultaneous eigenfunction of $\bar{D}$ and $\bar{E}$ associated to an irreducible spherical function with $P(1) = (a_0, \dots, a_\ell)^t \in \mathbb{C}^{\ell+1}$, then $a_0 = 1$. Therefore we have the following corollary about the correspondence between irreducible spherical functions and eigenfunctions of $\bar{D}$ and $\bar{E}$.

Corollary 7.3. *Given $\ell \in 2\mathbb{Z}_{\geq 0}$, every eigenfunction $P(u)$ of $\bar{D}$ and $\bar{E}$ analytic in $(0,1]$ with eigenvalues $\lambda = -n(n+2)$, $n \in \mathbb{Z}_{\geq 0}$, and μ, respectively, such that $P(1) = (1, a_1, \dots, a_\ell)^t$, corresponds to an irreducible spherical function Φ of type $\pi_\ell \in \hat{K}$. Conversely, every irreducible spherical function of type $\pi_\ell \in \hat{K}$ is associated to one of such functions $P(u)$.*

If we take $n = m_1 - m_2$ and $k = m_2 + \ell/2$ in Corollary 6.2 we have that for an eigenfunction $P(u)$ of D and E, associated to an irreducible spherical function of type $\pi_\ell \in \hat{K}$, the respective eigenvalues are of the form

$$\lambda = -n(n+2), \qquad \mu = -\frac{\ell}{2}\left(\frac{\ell}{2}+1\right) + \left(n + k - \frac{\ell}{2} + 1\right)\left(k - \frac{\ell}{2}\right),$$

with $0 \leq n$ and $0 \leq k \leq \min(n, \ell)$. And now we can state the main theorem of this paper.

Theorem 7.4. *There exists a one to one correspondence between the irreducible spherical functions of type $\pi_\ell \in \hat{K}$ and the vector valued polynomial functions $P(u) = (p_0(u), \dots, p_\ell(u))^t$ with*

$$p_j(u) = a_j\, {}_2F_1\left(\begin{smallmatrix} -n+j,\, n+j+2 \\ j+3/2 \end{smallmatrix}; \frac{1-u}{2}\right),$$

where $n \in \mathbb{Z}_{\geq 0}$, $a_0 = 1$ and $\{a_j\}_{j=0}^{\ell}$ satisfies the recursion relation

$$i\,\tfrac{j(\ell-j+1)(n-j+1)(n+j+1)}{2(2j-1)(2j+1)}\; a_{j-1} - \tfrac{j(j+1)}{2}\; a_j - i\tfrac{(j+1)(\ell+j+2)}{2}\; a_{j+1} = \mu\, a_j,$$

with μ of the form

$$\mu = -\frac{\ell}{2}\left(\frac{\ell}{2}+1\right) + \left(n + k - \frac{\ell}{2} + 1\right)\left(k - \frac{\ell}{2}\right),$$

for $k \in \mathbb{Z}$ such that $0 \leq k \leq \min(n, \ell)$.

7.2. **Reconstruction of an irreducible spherical function.**

Fixed $\ell \in 2\mathbb{Z}_{\geq 0}$ we know that a function $P = P(u)$ as in Theorem 7.4 is associated to a unique irreducible spherical function Φ of type $\pi_\ell \in \hat{K}$. Now we show how to construct explicitly the function Φ from such a P. Recall that P is a polynomial function.

Let us define the vector function $H(u) = UT(u)P(u) = (h_0(u), \dots, h_\ell(u))$, $u \in [-1, 1]$, with U and $T(u)$ as in Corollary 5.5 and let $diag\{H(u)\}$ denote the diagonal matrix valued function whose kk-entry is equal to the k-th entry of the vector valued function $H(u)$.

In the other hand, consider the function $H : S^3 \longrightarrow \mathrm{End}(V_\pi)$ associated to the irreducible spherical function Φ, from Corollary 5.5 and (35) we know that for $u \in (0,1)$

$$H(\sqrt{1-u^2}, 0, 0, u) = H(u).$$

Therefore, since both functions in the equality above are analytic in $(-1,1)$ and are continuous in $[-1,1]$, we have that

$$H(\sqrt{1-u^2}, 0, 0, u) = H(u),$$

for all $u \in [-1,1]$.

Since $H(kx) = \pi_\ell(k)H(x)\pi_\ell^{-1}(k)$ for every $x \in S^3$ and $k \in K$, we have found the explicit values of the function H on the sphere S^3. Then we can define the function $H : G \longrightarrow \mathrm{End}(V_{\pi_\ell})$ by

$$H(g) = H(gK), \qquad g \in G.$$

Finally, we have that the irreducible spherical function Φ is of the form

$$\Phi(g) = H(g)\Phi_{\pi_\ell}(g), \qquad g \in G,$$

where Φ_{π_ℓ} is the auxiliary spherical function introduced in Subsection 2.6.

8. Hypergeometrization

In this section for a fixed $\ell \in 2\mathbb{Z}_{\geq 0}$, we shall construct a sequence of matrix valued polynomials P_w closely related to irreducible spherical functions of type $\pi_\ell \in \hat{K}$. Then we will use the first matrix valued polynomial P_0 in the sequence to consider the differential operators $\widetilde{D} = {P_0}^{-1}\bar{D}P_0$ and $\widetilde{E} = {P_0}^{-1}\bar{E}P_0$.

For a given nonnegative integer w let us consider the matrix P_w whose k-th column is given by the vector valued function P associated to the spherical function $\Phi_\ell^{(w+\ell/2, -k+\ell/2)}$, for $k = 0,1,2,\dots,\ell$ (see Theorem 6.1). Therefore the k-th column of P_w is an eigenfunction of the operators $\bar{D}$ and $\bar{E}$ with eigenvalues

$$\lambda_w(k) = -(w+k)(w+k+2) \qquad \text{and} \qquad \mu_w(k) = w(\tfrac{\ell}{2} - k) - k(\tfrac{\ell}{2}+1),$$

respectively. Then, we have

$$[P_w(u)]_{j,k} = a_j^{w,k}\ {}_2F_1\left({{-w-k+j, w+k+j+2} \atop {j+3/2}} ; (1-u)/2 \right), \tag{41}$$

where $a_0^{w,k} = 1$ for all k and $\{a_j^{w,k}\}_{j=0}^{\ell}$ satisfies

$$\begin{aligned} & i\tfrac{j(\ell-j+1)(w+k-j+1)(w+k+j+1)}{2(2j-1)(2j+1)}\ a_{j-1}^{w,k} - \tfrac{j(j+1)}{2}\ a_j^{w,k} - i\tfrac{(j+1)(\ell+j+2)}{2}\ a_{j+1}^{w,k} \\ & = \big(w(\tfrac{\ell}{2}-k) - k(\tfrac{\ell}{2}+1)\big)\, a_j^{w,k} \end{aligned} \tag{42}$$

In the particular case that $w = 0$ we have the following explicit formulas for $a_j^{0,k}$.

Proposition 8.1. *We have*

$$\begin{aligned} a_j^{0,k} &= \frac{(-2i)^j k!\, j!}{(k-j)!(2j)!} \quad \textit{for } 0 \leq j \leq k \leq \ell, \\ a_j^{0,k} &= 0 \quad \textit{for } 0 \leq k < j \leq \ell. \end{aligned} \tag{43}$$

Proof. We only need to check that these $a_j^{0,k}$ satisfy the following three term recursive relation:

$$i\tfrac{j(\ell-j+1)(k-j+1)(k+j+1)}{2(2j-1)(2j+1)}a_{j-1}^{0,k}-\tfrac{j(j+1)}{2}a_j^{0,k}-i\tfrac{(j+1)(\ell+j+2)}{2}a_{j+1}^{0,k}=-\tfrac{k(\ell+2)}{2}a_j^{0,k}, \tag{44}$$

because $a_0^{0,k}=1$. Notice that if the coefficients $a_j^{0,k}$ are given by (43), for $0\le j\le k\le\ell$ we have

$$ia_{j-1}^{0,k}=-\tfrac{2j-1}{k-j+1}a_j^{0,k},\qquad ia_{j+1}^{0,k}=\tfrac{k-j}{2j+1}a_j^{0,k}.$$

Hence, for $0\le j\le k\le\ell$ (44) is equivalent to

$$-\tfrac{j(\ell-j+1)(k+j+1)}{2(2j+1)}a_j^{0,k}-\tfrac{j(j+1)}{2}a_j^{0,k}-\tfrac{(j+1)(\ell+j+2)(k-j)}{2(2j+1)}a_j^{0,k}=-\tfrac{k(\ell+2)}{2}a_j^{0,k},$$

which can be easily checked.
If $j=k+1$ we have

$$i\tfrac{(k+1)(\ell-(k+1)+1)(k-(k+1)+1)(k+(k+1)+1)}{2(2j-1)(2j+1)}\;a_k^{0,k}=0,$$

which is true. And if $j\ge k+2$ we just have $0=0$. Therefore the coefficients given by (43) satisfy (42) and the proof is finished. $\square$

Now we introduce the matrix valued function Ψ defined by the first "package" of spherical functions P_w with $w\ge0$, i.e.,

$$\Psi(u)=P_0(u). \tag{45}$$

Recall that $\Psi(u)=(\Psi_{jk})_{jk}$ is an upper triangular matrix polynomial with

$$\Psi_{jk}=\frac{(2j+1)(-2i)^j\,k!j!}{(k+j+1)!}C_{k-j}^{j+1}(u), \tag{46}$$

for $0\le j\le k\le\ell$, where $C_{k-j}^{j+1}(u)$ is the Gegenbauer polynomial

$$C_{k-j}^{j+1}(u)=\binom{k+j+1}{k-j}\,{}_2F_1\left(\begin{smallmatrix}-k+j\,,\,k+j+2\\ j+3/2\end{smallmatrix};(1-u)/2\right). \tag{47}$$

Since the k-th column of Ψ is an eigenfunction of $\bar D$ and $\bar E$ with eigenvalues $\lambda_0(k)=-k(k+2)$ and $\mu_0(k)=-k(\frac{\ell}{2}+1)$ respectively, the function Ψ satisfies

$$\bar D\Psi=\Psi\Lambda_0\quad\text{ and }\quad\bar E\Psi=\Psi M_0, \tag{48}$$

where $\Lambda_0=\sum_{k=0}^{\ell}\lambda_0(k)E_{kk}$ and $M_0=\sum_{k=0}^{\ell}\mu_0(k)E_{kk}$.

Remark 8.2. Since the entries of the diagonal of $\Psi(u)$ are nonzero constant polynomials we have that $\Psi(u)$ is invertible. Even more, the inverse $\Psi^{-1}(u)$ is also an upper triangular matrix polynomial. This can be easily checked, for instance, using the Crammer's rule, since the determinant of $\Psi(u)$ is a nonzero constant.

Theorem 8.3. *Let $\bar D$ and $\bar E$ be the differential operators defined in Theorem 5.1, and let Ψ the matrix valued function whose entries are given by* (46)*. Let $\widetilde D=\Psi^{-1}\bar D\Psi$ and $\widetilde E=\Psi^{-1}\bar E\Psi$, then*

$$\begin{aligned}\widetilde DF=&(1-u^2)F''+(-uC+S_1)F'+\Lambda_0F,\\ \widetilde EF=&(uR_2+R_1)F'+M_0F,\end{aligned}$$

for any C^∞-function F on $(0,1)$ with values in $\mathbb{C}^{\ell+1}$, where

$$C=\sum_{j=0}^{\ell}(2j+3)E_{jj}, \qquad S_1=\sum_{j=0}^{\ell}2(j+1)E_{j,j+1},$$

$$R_1=\sum_{j=0}^{\ell-1}\tfrac{(j+1)}{2}E_{j,j+1}-\sum_{j=0}^{\ell-1}\tfrac{\ell-j}{2}E_{j+1,j}, \qquad R_2=\sum_{j=0}^{\ell}(\tfrac{\ell}{2}-j)E_{j,j},$$

$$\Lambda_0=\sum_{j=0}^{\ell}-k(k+2)E_{j,j}, \qquad M_0=\sum_{j=1}^{\ell}-k(\tfrac{\ell}{2}+1)E_{j,j}.$$

Proof. By definition we have

$$\begin{aligned}\widetilde{D}\widetilde{F}=&(1-u^2)\widetilde{F}''+\Psi^{-1}[2(1-u^2)\Psi'-uC\Psi]\widetilde{F}'\\ &+\Psi^{-1}\left[(1-u^2)\Psi''-uC\Psi'-V\Psi\right]\widetilde{F},\\ \widetilde{E}\widetilde{F}=&\tfrac{i}{2}\Psi^{-1}[(1-u^2)Q_0+Q_1]\Psi\widetilde{F}'\\ &+\Psi^{-1}\left[\tfrac{i}{2}\left((1-u^2)Q_0+Q_1\right)\Psi'-\tfrac{i}{2}uM\Psi-\tfrac{1}{2}V_0\Psi\right]\widetilde{F}.\end{aligned}$$

By using (48) we observe that

$$\begin{aligned}(1-u^2)\Psi''-uC\Psi'-V\Psi=\bar{D}\Psi=\Psi\Lambda_0,\\ \tfrac{i}{2}\big((1-u^2)Q_0+Q_1\big)\Psi'-\tfrac{i}{2}uM\Psi-\tfrac{1}{2}V_0\Psi=\bar{E}\Psi=\Psi M_0.\end{aligned}$$

To complete the proof of this theorem, we use the following properties of the Gegenbauer polynomials (for the first three see [KS] page 40, and for the last one see [S]).

$$\frac{dC_n^\lambda}{du}(u)=2\lambda C_{n-1}^{\lambda+1}(u), \tag{49}$$

$$2(n+\lambda)uC_n^\lambda(u)=(n+1)C_{n+1}^\lambda(u)+(n+2\lambda-1)C_{n-1}^\lambda(u), \tag{50}$$

$$(1-u^2)\frac{dC_n^\lambda}{du}(u)+(1-2\lambda)uC_n^\lambda(u)=-\frac{(n+1)(2\lambda+n-1)}{2(\lambda-1)}C_{n+1}^{\lambda-1}(u), \tag{51}$$

$$\frac{(n+2\lambda-1)}{2(\lambda-1)}C_{n+1}^{\lambda-1}(u)=C_{n+1}^\lambda(u)-uC_n^\lambda(u). \tag{52}$$

We need to establish the following identities

$$[2(1-u^2)\Psi'-uC\Psi]=\Psi(-uC+S_1), \tag{53}$$

$$\tfrac{i}{2}[(1-u^2)Q_0+Q_1]\Psi=\Psi(uR_2+R_1). \tag{54}$$

Since (53) is a matrix identity, by looking at the jk-place we have

$$2(1-u^2)\Psi'_{jk}-uC_{jj}\Psi_{jk}=-\Psi_{jk}uC_{kk}+\Psi_{j,k-1}(S_1)_{k-1,k}.$$

Multiplying both sides by $\frac{(k+j+1)!}{(2j+1)(-2i)^j\,k!j!}$ and using (46) we have

$$2(1-u^2)\frac{d}{du}C_{k-j}^{j+1}-u(2j+3)C_{k-j}^{j+1}=-u(2k+3)C_{k-j}^{j+1}+2(k+j+1)C_{k-j-1}^{j+1}.$$

and by setting $\lambda=j+1$ and $n=k-j$ we get

$$2(1-u^2)\frac{dC_n^\lambda}{du}-u(2\lambda+1)C_n^\lambda=-u(2(n+\lambda)+1)C_n^\lambda+2(n+2\lambda-1)C_{n-1}^\lambda.$$

To see that this identity holds, we use (51) to write $\frac{dC_n^\lambda}{du}$ in terms of C_n^λ and $C_{n+1}^{\lambda-1}$, and (50) to express C_{n-1}^λ in terms of C_n^λ and C_{n+1}^λ. Finally we recognize the identity (52). Thus we have proved (53).

Now we need to verify the matrix identity (54). The jk-entry is given by (see (29) for the definition of the matrices Q_0 and Q_1)

$$\begin{aligned}\tfrac{i}{2}(1-u^2)(Q_0)_{j,j+1}\Psi_{j+1,k} + \tfrac{i}{2}(Q_1)_{j,j-1}\Psi_{j-1,k} = \\ u\Psi_{jk}(R_2)_{kk} + \Psi_{j,k+1}(R_1)_{k+1,k} + \Psi_{j,k-1}(R_1)_{k-1,k}.\end{aligned}$$

Again multiplying both sides by $\frac{(k+j+1)!}{(-2i)^j\, k!j!}$ and setting $\lambda = j+1$ and $n = k-j$, we obtain

$$\begin{aligned}(1-u^2)\frac{\lambda^2(\ell+\lambda+1)}{n+2\lambda}C_{n-1}^{\lambda+1} - \frac{(\ell-\lambda+2)(n+2\lambda-1)}{4}C_{n+1}^{\lambda-1} = \\ u\left(\tfrac{\ell}{2}-(n+\lambda-1)\right)(2\lambda-1)C_n^\lambda - \frac{(\ell-n-\lambda+1)(2\lambda-1)(n+\lambda)}{2(n+2\lambda)}C_{n+1}^\lambda \\ + \frac{(2\lambda-1)(n+2\lambda-1)}{2}C_{n-1}^\lambda.\end{aligned}$$

Now we first use (51) combined with (49) to write $C_{n-1}^{\lambda+1}$ in terms of C_n^λ and $C_{n+1}^{\lambda-1}$, and then we use (50) to express C_{n-1}^λ in terms of C_n^λ and C_{n+1}^λ. Then we get

$$\tfrac{(\lambda^2-\ell\lambda-3\lambda-\ell n-2n)(2\lambda-1)}{2(n+2\lambda)}\left(uC_n^\lambda(u) + \frac{(n+2\lambda-1)}{2(\lambda-1)}C_{n+1}^{\lambda-1}(u) - C_{n+1}^\lambda(u)\right) = 0,$$

which is true by (52).

□

9. Orthogonal Polynomials

The aim of this section is to build classical sequences of matrix valued orthogonal polynomials from our previous work. This means to exhibit a weight matrix W supported on the real line, a sequence $(\widetilde{P}_w)_{w\geq 0}$ of matrix polynomials such that $\deg(\widetilde{P}_w) = w$ with the leading coefficient of $\widetilde{P}_w$ nonsingular, orthogonal with respect to W, and a second order (symmetric) differential operator $\widetilde{D}$ such that $\widetilde{D}\widetilde{P}_w = \widetilde{P}_w\Lambda_w$ where Λ_w is a real diagonal matrix. Moreover we point out that we also have a first order (symmetric) differential operator $\widetilde{E}$ such that $\widetilde{E}\widetilde{P}_w = \widetilde{P}_w M_w$, where M_w is a real diagonal matrix.

From $\widetilde{D}$ (see Theorem 8.3) we obtain a new differential operator D by making the change of variables $s = (1-u)/2$. Thus

$$DF = s(1-s)F'' - \left(\frac{S_1 - C}{2} + sC\right)F' + \Lambda_0 F.$$

9.1. **Polynomial solutions of $DF = \lambda F$.**

We start by studying the $\mathbb{C}^{\ell+1}$-vector valued polynomials $F = F(u)$ such that

$$s(1-s)F'' + (B - sC)F' + (\Lambda_0 - \lambda)F = \lambda F, \tag{55}$$

where

$$B = \frac{C - S_1}{2},$$

and C, S_1 and Λ_0 are those matrices given in Theorem 8.3. This equation is an instance of a matrix hypergeometric differential equation studied in [Tir03]. Since the eigenvalues of B are not in $-\mathbb{N}_0$ the function F is determined by $F_0 = F(0)$. For $|u| < 1$ it is given by

$$F(u) = {}_2H_1\left({}^{C,-\Lambda_0+\lambda}_{\;\;B} ; s\right) F_0 = \sum_{j=0}^{\infty} \frac{u^j}{j!}[B; C; -\Lambda_0 + \lambda]_j F_0, \qquad F_0 \in \mathbb{C}^{\ell+1}, \tag{56}$$

where the symbol $[B; C; -\Lambda_0 + \lambda]_j$ is defined inductively by

$$\begin{aligned} &[B; C; -\Lambda_0 + \lambda]_0 = 1, \\ &[B; C; -\Lambda_0 + \lambda]_{j+1} = \\ &\qquad (B + j)^{-1}(j(C + j - 1) - \Lambda_0 + \lambda)[B; C; -\Lambda_0 + \lambda]_j, \end{aligned}$$

for all $j \geq 0$.

Therefore there exists a polynomial solution of (55) if and only if the coefficient $[B; C; -\Lambda_0 + \lambda]_j$ is a singular matrix for some $j \in \mathbb{Z}_{\geq 0}$. Since the matrix $B + j$ is invertible for any $j \in \mathbb{Z}$, we have that there is a polynomial solution of degree w of (55) if and only if there exists $F_0 \in \mathbb{C}^{\ell+1}$ such that $[B; C; -\Lambda_0 + \lambda]_w F_0 \neq 0$ and $(w(C + w - 1) - \Lambda_0 + \lambda)[B; C; -\Lambda_0 + \lambda]_w F_0 = 0$. The matrix

$$M_w = w(C + w - 1) - \Lambda_0 + \lambda = \sum_{j=0}^{\ell}((j + w)(j + w + 2) + \lambda)E_{jj}$$

is diagonal, and M_w is a singular matrix if and only if λ is of the form $-(k+w)(k+w+2)$ for some k such that $0 \leq k \leq \ell$. When this happens the kernel of M_w is the subspace generated by e_k (the vector of the canonical basis $\{e_0, e_1, \ldots, e_\ell\}$ of $\mathbb{C}^{\ell+1}$). Observe that for every j the subspace generated by $\{e_0, e_1, \ldots, e_k\}$ is invariant by $(B - j)^{-1}$ because it is upper triangular, and that for $j < w$, M_j is a diagonal matrix whose first $k + 1$ entries are not zero. Therefore there exists F_0 such that $[B; C; -\Lambda_0 + \lambda]_w F_0 = e_k$. Then

$$F(u) = {}_2H_1\left({}^{C,-\Lambda_0+\lambda}_{\;\;B} ; u\right) F_0$$

is a vector polynomial of degree w. Even more, the k-th entry of the solution is a polynomial of degree w, and all the other entries are of lower degrees. Hence we have the next propositions.

Proposition 9.1. *Given $\lambda \in \mathbb{C}$, the equation $DF = \lambda F$ has a polynomial solution if and only if λ is of the form $-n(n + 2)$ for $n \in \mathbb{N}_0$. When this happens, for every $w \geq 0$ such that $n - \ell \leq w \leq n$ there is a polynomial solution of degree w, and every polynomial solution is a linear combination of these ones.*

Proposition 9.2. *Given $w \geq 0$ there exist exactly $\ell+1$ values of λ such that $DF = \lambda F$ has a polynomial solution of degree w, and for each one of these λ's there is only one polynomial solution of degree w up to a scalar and modulo polynomials of lower degree. These possible values of λ are*

$$\lambda_w(k) = -(k + w)(k + w + 2)$$

for $0 \leq k \leq \ell$, and the polynomial of degree w corresponding to the eigenvalue $\lambda_w(k)$ has its k-th entry of degree w, and all the other entries are of lower degrees.

9.2. **The Inner Product.**

Now, given a finite dimensional irreducible representation $\pi = \pi_\ell$ of K in the vector space V_π, let $(C(G)\otimes \mathrm{End}(V_\pi))^{K\times K}$ be the space of all continuous functions $\Phi : G \longrightarrow \mathrm{End}(V_\pi)$ such that $\Phi(k_1 g k_2) = \pi(k_1)\Phi(g)\pi(k_2)$ for all $g \in G$, $k_1, k_2 \in K$. Let us equip V_π with an inner product such that $\pi(k)$ becomes unitary for all $k \in K$. Then we introduce an inner product in the vector space $(C(G)\otimes \mathrm{End}(V_\pi))^{K\times K}$ by defining

$$\langle \Phi_1, \Phi_2\rangle = \int_G \mathrm{tr}(\Phi_1(g)\Phi_2(g)^*)\, dg\,, \tag{57}$$

where dg denotes the Haar measure of G normalized by $\int_G dg = 1$, and $\Phi_2(g)^*$ denotes the adjoint of $\Phi_2(g)$ with respect to the inner product in V_π.

Let us write $\Phi_1 = H_1\Phi_\pi$, $\Phi_2 = H_2\Phi_\pi$ as we did in (7), and put $H_1(u) = (h_0(u), \cdots, h_\ell(u))^t$, $H_2(u) = (f_0(u), \cdots, f_\ell(u))^t$ as we did in Subsection 7.2.

Proposition 9.3. *If $\Phi_1, \Phi_2 \in (C(G)\otimes \mathrm{End}(V_\pi))^{K\times K}$ then*

$$\langle \Phi_1, \Phi_2\rangle = \frac{2}{\pi}\int_{-1}^{1}\sqrt{1-u^2}\sum_{i=0}^{\ell} h_j(u)\overline{f_j(u)}\, du.$$

Proof. Let us consider the element $E_1 = E_{14} - E_{41} \in \mathfrak{g}$. Then, as $\mathfrak{so}(4)_{\mathbb{C}} \simeq \mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$, ad E_1 has 0 and $\pm i$ as eigenvalues with multiplicity 2.

Let $A = \exp \mathbb{R}E_1$ be the Lie subgroup of G of all elements of the form

$$a(t) = \exp tE_1 = \begin{pmatrix} \cos t & 0 & 0 & \sin t\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ -\sin t & 0 & 0 & \cos t\end{pmatrix}, \qquad t\in\mathbb{R}.$$

Now Theorem 5.10, page 190 in [Hel84] establishes that for every $f \in C(G/K)$ and a suitable c_*

$$\int_{G/K} f(gK)\, dg_K = c_*\int_{K/M}\Big(\int_{-\pi}^{\pi}\delta_*(a(t))f(ka(t)K)\, dt\Big)\, dk_M\,,$$

where the function $\delta_* : A \longrightarrow \mathbb{R}$ is defined by

$$\delta_*(a(t)) = \prod_{\nu\in\Sigma^+}|\sin it\nu(E_1)|,$$

and dg_K and dk_M are respectively the left invariant measures on G/K and K/M normalized by $\int_{G/K} dg_K = \int_{K/M} dk_M = 1$. In our case we have $\delta_*(a(t)) = \sin^2 t$.

Since the function $g \mapsto \mathrm{tr}(\Phi_1(g)\Phi_2(g)^*)$ is invariant under left and right multiplication by elements in K, we have

$$\langle\Phi_1,\Phi_2\rangle = c_*\int_{-\pi}^{\pi}\sin^2 t\ \mathrm{tr}\left(\Phi_1(a(t))\Phi_2(a(t))^*\right)\, dt. \tag{58}$$

Also for each $t\in[-\pi,0]$ we have that $(I-2(E_{11}+E_{22}))a(t)(I-2(E_{11}+E_{22})) = a(-t)$, with $I-2(E_{11}+E_{22})$ in K. Then we have

$$\langle\Phi_1,\Phi_2\rangle = 2c_*\int_{0}^{\pi}\sin^2 t\ \mathrm{tr}\left(\Phi_1(a(t))\Phi_2(a(t))^*\right)\, dt. \tag{59}$$

By the definition of the auxiliary function $\Phi_\pi(g)$ (see Subsection 2.6) we have that $\Phi_1(a(t))\Phi_2(a(t))^* = H_1(a(t))H_2(a(t))^*$. Therefore, making the change of variables $\cos(t) = u$, we have

$$\langle \Phi_1, \Phi_2 \rangle = 2c_* \int_{-1}^{1} \sqrt{1-u^2} \sum_{j=0}^{\ell} h_j(u)\overline{f_j(u)} du.$$

To find the value of c_* we consider the trivial case $\Phi_1 = \Phi_2 = I$, having so from (57) and (58)

$$\ell + 1 = c_* \int_{-\pi}^{\pi} \sin^2 t\,(\ell+1)\,dt,$$

then we get $c_* = \pi^{-1}$ and the proposition follows. □

9.3. **Our sequence of matrix orthogonal polynomials.**

We take $\widetilde{P}_w = \Psi^{-1} P_w$ where Ψ is the matrix polynomial introduced in (45) and P_w are the matrix polynomials defined in (41). The function Ψ^{-1} is also a matrix polynomial as we observed in Remark 8.2, then so is $\widetilde{P}_w$.

Proposition 9.4. *The columns $\{\widetilde{P}_w^k\}_{k=0,\dots,\ell}$ of $\widetilde{P}_w$ are $\mathbb{C}^{\ell+1}$-valued polynomials of degree w. Even more, for $j > k$ the degree of $\left(\widetilde{P}_w^k\right)_j$ is lower than w, and for $j = k$ the degree of $\left(\widetilde{P}_w^k\right)_j$ is w.*

Proof. Consider $\lambda = -n(n+2)$, $n \geq 0$. Then $\widetilde{P}_w^k = \Psi^{-1} P_w^k$ are polynomials solutions of $\widetilde{D}\widetilde{F} = \lambda \widetilde{F}$, for (w,k) such that $w + k = n$. But in the Proposition 9.1 we have established that the degrees of the polynomial solutions are $\{n-\ell, \dots, n\}$, then, if w' is the degree of $\widetilde{P}_w^k$ we have that $n - \ell \leq w' \leq \ell$. In the other hand $\widetilde{P}_w^k$ is a solution for $\widetilde{E}\widetilde{F} = \mu_w(k)\widetilde{F}$, where $\mu_w(k) = w(\frac{\ell}{2} - k) - k(\frac{\ell}{2} + 1)$. Hence we can write $\widetilde{P}_w^k = \sum_{j=0}^{w'} u^j A_j$ with $A_j \in \mathbb{C}^{\ell+1}$ and we have

$$(uR_2 + R_1) \sum_{j=1}^{w'} j u^{j-1} A_j + (M_0 - \mu_w(k)) \sum_{j=0}^{w'} u^j A_j = 0,$$

then

$$\sum_{j=0}^{w'} \left[(jR_2 A_j + (j+1) R_1 A_{j+1} + (M_0 - \mu_w(k)) A_j \right] u^j = 0,$$

denoting $A_{w'+1} = 0$. In particular we have that

$$w' R_2 A_{w'} + (M_0 - \mu_w(k)) A_{w'} = 0,$$

and if we look at the $(n - w')$-th entry of this matrix equation, we observe that

$$w'(\tfrac{\ell}{2} - (n - w')) - (n - w')(\tfrac{\ell}{2} + 1) - \mu_w(k) = 0,$$

because $(A_{w'})_{n-w'} \neq 0$ (see Proposition 9.2). Then

$$\mu_{w'}(n - w') = \mu_w(n - w),$$

and necessarily $w = w'$ because of $\mu_w(n-w)$ as a function on $w \in \{n-\ell, \dots, n\}$ is injective. Therefore, $\widetilde{P}_w^k$ is of degree w. The second assertion follows from Proposition 9.2.

□

Consider now the sequence of matrix polynomials $\{\widetilde{P}_w(u)\}_{w\geq 0}$. For each $\widetilde{P}_w(u)$ the k-th column is given by a vector $\widetilde{P}^k_w(u)$ associated to the spherical function $\Phi_\ell^{(w-\ell/2,-k+\ell/2)}$.

Recall that $\Phi_\ell^{(w-\ell/2,-k+\ell/2)} = UT(u)\Psi\widetilde{P}^k_w(u)\Phi_\pi(u)$, then Proposition 9.3 tells us that

$$\langle\Phi_\ell^{(w-\ell/2,-k+\ell/2)}, \Phi_\ell^{(w'-\ell/2,-k'+\ell/2)}\rangle = \frac{2}{\pi}\int_{-1}^{1}\sqrt{1-u^2}[UT(u)\Psi\widetilde{P}^k_w(u)]^*[UT(u)\Psi\widetilde{P}^{k'}_{w'}(u)]\,dr,$$

hence,

$$\int_{-1}^{1}\widetilde{P}^k_w(u)^*W(u)\widetilde{P}^{k'}_{w'}(u)\,dr = \langle\Phi_\ell^{(w-\ell/2,-k+\ell/2)}, \Phi_\ell^{(w'-\ell/2,-k'+\ell/2)}\rangle,$$

where

$$W(u) = \frac{2}{\pi}\sqrt{1-u^2}\Psi^*T^*(u)U^*UT(u)\Psi. \tag{60}$$

Therefore $\widetilde{P}^k_w$ and $\widetilde{P}^{k'}_{w'}$ are orthogonal with respect to W if $(w,k)\neq(w',k')$, since using Schur's orthogonality relations it follows that non equivalent spherical functions are orthogonal with respect to the inner product (57).

Theorem 9.5. *The sequence $(\widetilde{P}_w)_{w\geq 0}$ is an orthogonal sequence of matrix valued polynomials with respect to W, such that*

$$\widetilde{D}\widetilde{P}_w = \widetilde{P}_w\Lambda_w \qquad \text{and} \qquad \widetilde{E}\widetilde{P}_w = \widetilde{P}_w M_w,$$

where $\Lambda_w = \sum_{k=0}^{\ell}\lambda_w(k)E_{kk}$, and $M_w = \sum_{k=0}^{\ell}\mu_w(k)E_{kk}$, with

$$\lambda_w(k) = -(w+k)(w+k+2) \qquad \text{and} \qquad \mu_w(k) = w(\tfrac{\ell}{2}-k) - k(\tfrac{\ell}{2}+1).$$

Proof. Given w and w', non negative integers we have

$$\begin{aligned}\langle\widetilde{P}_w,\widetilde{P}_{w'}\rangle = \int_{-1}^{1}\widetilde{P}_w(u)^*W(u)\widetilde{P}_{w'}(u)\,dr &= \sum_{k,k'=0}^{\ell}E_{k,k'}\int_{-1}^{1}\widetilde{P}^k_w(u)^*W(u)\widetilde{P}^{k'}_{w'}(u)\,dr\\ &= \sum_{k,k'=0}^{\ell}E_{k,k'}\delta_{w,w'}\delta_{k,k'}\int_{-1}^{1}\widetilde{P}^k_w(u)^*W(u)\widetilde{P}^{k'}_{w'}(u)\,dr\\ &= \delta_{w,w'}\sum_{k=0}^{\ell}E_{k,k}\int_{-1}^{1}\widetilde{P}^k_w(u)^*W(u)\widetilde{P}^{k}_{w'}(u)\,dr,\end{aligned}$$

then orthogonality is proved. As we have established in Proposition 9.4 $\widetilde{P}^k_w(u)$ is a polynomial of degree w, and for $j>k$ the entries $(\widetilde{P}^k_w(u))_j$ are of lower degrees, therefore $\widetilde{P}_w(u)$ has a non singular leading coefficient. For the last assertion recall that each column $\widetilde{P}^k_w(u)$ is an eigenfunction of the operator $\widetilde{D}$ and $\widetilde{E}$ introduced in Theorem 8.3, with respective eigenvalues $\lambda_w(k) = -(w+k)(w+k+2)$ and $\mu_w(k) = w(\frac{\ell}{2}-k) - k(\frac{\ell}{2}+1)$. □

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